Four Reflections
The Road to a Complete Unification of Physics

© 2019 Copyright Ty K Coburn

Until and unless other notice is given, copies of material found herein may only be made by students, instructors, and others solely for their personal use. To include any original material, or original presentation of known material, contained herein in a publication, written prior approval from Ty K Coburn, or his heirs, is required.

All Rights Reserved. No part of this book may be reproduced, stored in a retrieval system, transmitted or translated into machine language, in any form or by any means, electronic, mechanical, photocopying, recording or otherwise, without the prior written permission of the author, his heirs if he is no longer alive, or the publisher, except for brief quotations embodied in research articles and reviews. Any permissible public use must cite this book as a source.
BOOK IV: The Fourth Reflection

IMAGE THEORY

First Addition
# Table of Contents

Chapter 1 .......................................................................................................................... 10

What is Logic? .................................................................................................................... 10

1.0 Introduction .................................................................................................................. 10

1.0.1 Reductionism vs. Non-Reductionism ................................................................. 10

1.0.2 Relativistic vs. Quantum Mathematics ............................................................. 11

1.0.2.1 The Foundations of the Mathematics (Relativity and Quantum) ................. 11

1.1 The Philosophy of Logic .............................................................................................. 12

1.1.1 Logical Realism .................................................................................................... 12

1.1.2 Logical Nominalism ............................................................................................. 13

1.1.3 Nominalism vs. Realism in Logic ........................................................................ 14

1.1.4 The Problem of a Single Set System of Logic ..................................................... 15

1.1.4.1 Ideas vs. Sentences ....................................................................................... 15

1.1.4.2 The Trouble with Universals ....................................................................... 16

1.2 Concluding Remarks ................................................................................................. 17

Chapter 2 .......................................................................................................................... 18

The Theory of Objective Logic ......................................................................................... 18

2.0 Introduction .................................................................................................................. 18

2.1 The Foundations of Logical Systems .......................................................................... 18

2.2 The Syntax of Objective Logic .................................................................................. 18

2.2.1 The Object Language .......................................................................................... 18

2.2.2 Statements (the Semantics of Objective Logic) .................................................. 19

2.2.3 Systems ................................................................................................................ 19

2.2.3.1 Valid Systems ............................................................................................... 19

2.2.3.2 Systems of Systems ...................................................................................... 20

2.2.3.3 Consistent and Inconsistent Systems .......................................................... 20

2.2.3.4 Complete and Incomplete Systems .............................................................. 21

2.2.3.5 Equal and Equivalent Systems .................................................................... 21

2.3 Valid, Invalid and Partially Valid Complete and Consistent Systems ..................... 22

2.3.1 Valid Systems ...................................................................................................... 22

2.3.2 Complete and Consistent Invalid Systems ....................................................... 22

2.3.3 Complete and Consistent Partially Valid Systems .............................................. 22

2.3.4 Proposition on Complete and Consistent Systems ............................................ 22
3.3 The Rational Numbers

3.2 The Integers

3.0 Introduction

Chapter 3

2.11 Concluding Remarks

2.10 Paradoxes

2.9 The Laws of Valid Systems

2.8 Validly Ordered Systems

2.7 Relations

2.6 Functions on Systems

2.5 Set Operations on Valid Systems

2.4 Characteristics of a System

2.4.3 Universal Systems

2.4.2 Propositions on Systems

2.4.1 Subsystems

2.4 Characteristics of a System

2.3.2 Properties of the Integers

2.3.1 Instantiating the Integers

2.2 Valid Functions

2.1 Characteristics of a System

2.1.2 Properties of the Natural Numbers

2.1.1 Instantiation of the Natural Numbers

2.1 Characteristics of a System

2.0 Introduction

3.0 Introduction

3.1 The Natural Numbers

3.2 The Integers

3.3 The Rational Numbers

4

45
3.3.1 Instantiating the Rational Numbers ................................................................. 45
3.3.2 Properties of the Rational Numbers ................................................................. 47
3.4 The Real Numbers ................................................................................................. 48
3.5 Vector Spaces as Valid Spaces ............................................................................... 49
  3.5.1 Hilbert Spaces .................................................................................................. 49
3.6 Establishing the Complex Plane ............................................................................. 50
  3.6.1 Complex Numbers ............................................................................................ 51
    3.6.1.1 Establishing a Metric in Objective Space .................................................. 51
  3.6.2 Objective Complex Numbers ........................................................................... 52
    3.6.2.1 Number Interference .................................................................................. 53
3.7 Anomalies within Instantiation Space .................................................................... 56
  3.7.1 Operations on Complex Numbers in Objective Space .................................... 57
3.8 Concluding Remarks .............................................................................................. 58
Chapter 4 .................................................................................................................... 59
Complex Analysis in Instantiation Space .................................................................... 59
4.0 Introduction ............................................................................................................. 59
4.1 Instantiated Complex Numbers ............................................................................. 59
  4.1.1 Discontinuities in Instantiation Space ............................................................. 59
  4.1.2 Guaranteeing Unique Instantiations ............................................................... 63
4.2 Projection ................................................................................................................ 64
  4.2.1 The Modulus of \(z\) and \(\arg z\) ........................................................................ 64
  4.2.2 The Product of Complex Numbers in Instantiation Space ............................... 65
    4.2.2.1 De Moivre’s Theorem Instantiated ............................................................ 66
  4.2.3 The \(n\)th Roots of \(z\) Instantiated ........................................................................ 66
    4.2.3.1 The \(n\)th Roots of Unity Instantiated ......................................................... 68
  4.2.4 The Dot Product Instantiated ........................................................................... 69
  4.2.5 The Cross Product Instantiated ....................................................................... 70
4.3 Functions of a Complex Variable in Instantiation Space ...................................... 70
  4.3.1 Multivalued Functions in Instantiation Space .................................................. 72
    4.3.1.1 Branch Points ......................................................................................... 72
4.4 Complex Differentiation in Instantiation Space .................................................... 73
  4.4.1 The Cauchy-Riemann Equations Instantiated .................................................. 73
  4.4.2 The Jacobian .................................................................................................... 74
    4.4.2.1 The Derivative of a Function in Instantiation Space ................................. 76
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.4.2.2 The Gradient Instantiated</td>
<td>78</td>
</tr>
<tr>
<td>4.4.2.3 The Divergence Instantiated</td>
<td>79</td>
</tr>
<tr>
<td>4.4.2.4 The Curl Instantiated</td>
<td>79</td>
</tr>
<tr>
<td>4.4.2.5 The Laplacian Instantiated</td>
<td>80</td>
</tr>
<tr>
<td>4.5 Complex Curves in Instantiation Space</td>
<td>80</td>
</tr>
<tr>
<td>4.6 Conformal Mappings</td>
<td>81</td>
</tr>
<tr>
<td>4.6.1 Complex Integration in Instantiation Space</td>
<td>82</td>
</tr>
<tr>
<td>4.6.1.1 The Connection between Real and Complex Line Integrals</td>
<td>83</td>
</tr>
<tr>
<td>4.7 The Length of a Line in Instantiation Space</td>
<td>84</td>
</tr>
<tr>
<td>4.8 Some Observations</td>
<td>85</td>
</tr>
<tr>
<td>4.9 Concluding Remarks</td>
<td>88</td>
</tr>
<tr>
<td>Chapter 5</td>
<td>91</td>
</tr>
<tr>
<td>Objective Physics</td>
<td>91</td>
</tr>
<tr>
<td>5.0 Introduction</td>
<td>91</td>
</tr>
<tr>
<td>5.1 Objective Mathematics Interpreted</td>
<td>91</td>
</tr>
<tr>
<td>5.1.1 Faithful and Unfaithful Information</td>
<td>92</td>
</tr>
<tr>
<td>5.1.2 The Mind, Body and the External World</td>
<td>92</td>
</tr>
<tr>
<td>5.1.3 What is an Observation?</td>
<td>93</td>
</tr>
<tr>
<td>5.1.3.1 Information Interference</td>
<td>93</td>
</tr>
<tr>
<td>5.1.4 A Summary of Objective Mathematics Interpreted</td>
<td>93</td>
</tr>
<tr>
<td>5.2 Mathematical Preliminaries – the Fundamental Function</td>
<td>94</td>
</tr>
<tr>
<td>5.2.1 The Exponential Inverse of the Fundamental Function</td>
<td>95</td>
</tr>
<tr>
<td>5.2.2 The Square Root of the Fundamental Function</td>
<td>96</td>
</tr>
<tr>
<td>5.2.3 Differentiating the Fundamental Function</td>
<td>97</td>
</tr>
<tr>
<td>5.2.3.1 Differentiating the Inverse of the Fundamental Function</td>
<td>97</td>
</tr>
<tr>
<td>5.2.4 Integrating the Fundamental Function</td>
<td>98</td>
</tr>
<tr>
<td>5.2.4.1 Integrating the Exponential Inverse of the Fundamental Function</td>
<td>99</td>
</tr>
<tr>
<td>5.2.5 Relationship between the Derivative and Integral</td>
<td>99</td>
</tr>
<tr>
<td>5.3 The Physics of Image Theory</td>
<td>100</td>
</tr>
<tr>
<td>5.3.1 The Principle of Equivalence within Image Theory</td>
<td>100</td>
</tr>
<tr>
<td>5.3.2 Deriving the Speed of Light</td>
<td>102</td>
</tr>
<tr>
<td>5.3.2.1 The Difference Between Dark- and Light-Light Limit Speeds</td>
<td>104</td>
</tr>
<tr>
<td>5.3.3 Inverse Acceleration</td>
<td>104</td>
</tr>
<tr>
<td>5.3.3.1 Renormalizing the Inverse of the Fundamental Function</td>
<td>105</td>
</tr>
<tr>
<td>Section</td>
<td>Page</td>
</tr>
<tr>
<td>------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>7.0 Introduction</td>
<td>147</td>
</tr>
<tr>
<td>7.0.1 Tensors as Instantiations</td>
<td>147</td>
</tr>
<tr>
<td>7.0.1.1 Valid Instantiations of Coordinates</td>
<td>149</td>
</tr>
<tr>
<td>7.0.2 Higher Order Tensors</td>
<td>150</td>
</tr>
<tr>
<td>7.1 The Metric Tensor and Number Interference</td>
<td>151</td>
</tr>
<tr>
<td>7.1.1 The Generalization of Number Interference</td>
<td>152</td>
</tr>
<tr>
<td>7.1.2 The Metric</td>
<td>153</td>
</tr>
<tr>
<td>7.1.2.1 When the Elements of the $g_{ij}$ are Functions of the Coordinates</td>
<td>154</td>
</tr>
<tr>
<td>7.1.3 The $g_{ij}$’s and Commutation</td>
<td>155</td>
</tr>
<tr>
<td>7.2 Proper-Time</td>
<td>156</td>
</tr>
<tr>
<td>7.2.1 Dark- and Light- Proper-Time</td>
<td>156</td>
</tr>
<tr>
<td>7.2.2 Dark- and Light- Generalized Proper-Time</td>
<td>157</td>
</tr>
<tr>
<td>7.3 Geodesics in Intrinsic Geometry</td>
<td>158</td>
</tr>
<tr>
<td>7.3.1 The Covariant Derivative of the Tangent Vector</td>
<td>160</td>
</tr>
<tr>
<td>7.4 Curvature and the Parallel Transport of a Vector</td>
<td>161</td>
</tr>
<tr>
<td>7.4.1 Curved vs. Flat Spaces</td>
<td>161</td>
</tr>
<tr>
<td>7.4.2 Computing Curvature</td>
<td>162</td>
</tr>
<tr>
<td>7.4.3 The Curvature Tensor</td>
<td>162</td>
</tr>
<tr>
<td>7.4.3.1 Commutation and Curvature</td>
<td>162</td>
</tr>
<tr>
<td>7.4.3.2 Curvature and the Ricci Tensor</td>
<td>165</td>
</tr>
<tr>
<td>7.4.3.3 The Curvature Scalar</td>
<td>165</td>
</tr>
<tr>
<td>7.5 Einstein’s Search for a Gravitational Tensor Equation</td>
<td>165</td>
</tr>
<tr>
<td>7.5.1 The Cosmological Constant</td>
<td>166</td>
</tr>
<tr>
<td>7.6 Einstein’s Equations Due to Intrinsic Time</td>
<td>166</td>
</tr>
<tr>
<td>7.6.1 Intrinsic Time</td>
<td>167</td>
</tr>
<tr>
<td>7.6.2 Einstein’s Equations Modified</td>
<td>168</td>
</tr>
<tr>
<td>7.6.3 Elimination of the Cosmological Constant</td>
<td>170</td>
</tr>
<tr>
<td>7.6.4 Conjecture on the Physics at Small-Scales</td>
<td>170</td>
</tr>
<tr>
<td>7.7 How to Build a Black Hole</td>
<td>171</td>
</tr>
<tr>
<td>7.8 Concluding Remarks</td>
<td>172</td>
</tr>
<tr>
<td>Chapter 8</td>
<td>174</td>
</tr>
<tr>
<td>Image Theory and Quantum Mechanics</td>
<td>174</td>
</tr>
<tr>
<td>8.0 Introduction</td>
<td>174</td>
</tr>
<tr>
<td>Section</td>
<td>Page</td>
</tr>
<tr>
<td>------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>8.1 Schrödinger’s Equation Revisited</td>
<td>175</td>
</tr>
<tr>
<td>8.1.1 Simple Harmonic Motion (Vibration)</td>
<td>175</td>
</tr>
<tr>
<td>8.1.2 Wave Motion</td>
<td>175</td>
</tr>
<tr>
<td>8.1.3 The Wave Equation</td>
<td>176</td>
</tr>
<tr>
<td>8.1.3.1 Solution to the Wave Equation</td>
<td>176</td>
</tr>
<tr>
<td>8.1.4 The Derivation of Schrödinger’s Equation</td>
<td>176</td>
</tr>
<tr>
<td>8.2 A Facsimile of Schrödinger’s Equation within Image Theory</td>
<td>177</td>
</tr>
<tr>
<td>8.3 Instantiations and Quantum Mechanics</td>
<td>181</td>
</tr>
<tr>
<td>8.3.1 The Dirac δ-Function</td>
<td>181</td>
</tr>
<tr>
<td>8.3.2 Position Instantiation</td>
<td>182</td>
</tr>
<tr>
<td>8.3.2.1 The Physical Interpretation of Position Instantiation</td>
<td>183</td>
</tr>
<tr>
<td>8.3.3 Momentum Instantiation</td>
<td>183</td>
</tr>
<tr>
<td>8.3.3.1 The Physical Interpretation of the Instantiation of Momentum</td>
<td>185</td>
</tr>
<tr>
<td>8.4 Deriving a Substitute for Schrödinger’s Equation</td>
<td>185</td>
</tr>
<tr>
<td>8.4.1 The Interpretation of $\Psi$</td>
<td>186</td>
</tr>
<tr>
<td>8.4.2 Determining Position and Momentum Simultaneously</td>
<td>187</td>
</tr>
<tr>
<td>8.4.3 Energy Levels in Image Theory</td>
<td>187</td>
</tr>
<tr>
<td>8.5 Image Theory and the Quantum Field</td>
<td>188</td>
</tr>
<tr>
<td>8.5.1 The Quantum Field in Image Theory</td>
<td>189</td>
</tr>
<tr>
<td>8.6 Concluding Remarks</td>
<td>191</td>
</tr>
<tr>
<td>Chapter 9</td>
<td>193</td>
</tr>
<tr>
<td><strong>EPILOGUE</strong></td>
<td>193</td>
</tr>
<tr>
<td>9.0 Introduction</td>
<td>193</td>
</tr>
<tr>
<td>9.1 The Physics of the Big and the Small</td>
<td>195</td>
</tr>
<tr>
<td>9.1.1 Small-Scale Physics</td>
<td>195</td>
</tr>
<tr>
<td>9.1.2 Large-Scale Physics</td>
<td>195</td>
</tr>
<tr>
<td>9.1.3 Combining the Small with the Large</td>
<td>196</td>
</tr>
<tr>
<td>9.1.3.1 A Tensor Equation for the Small-Scale</td>
<td>198</td>
</tr>
<tr>
<td>9.2 The Time Epoch of the Universe</td>
<td>201</td>
</tr>
<tr>
<td>9.2.1 The Wave Length of the Universe</td>
<td>201</td>
</tr>
<tr>
<td>9.3 Deriving the Wave Function of the Universe</td>
<td>203</td>
</tr>
<tr>
<td>9.3.1 The Energy Interpretation of the Wave Function</td>
<td>204</td>
</tr>
<tr>
<td>9.4 Is the Universe Finite or Infinite?</td>
<td>207</td>
</tr>
</tbody>
</table>
Chapter 1

What is Logic?

“There exists, if I am not mistaken, an entire world which is the totality of mathematical truths, to which we have access only with our mind, just as a world of physical reality exists, the one like the other independent of ourselves, both of divine creation.”

– Charles Hermite

1.0 Introduction

The first three books in the series examined the difficulties encountered in finding a way to unite the theory of relativity and quantum mechanics into a single consistent theory. The first book, “The Problem of Unification”, explores the various attempts at developing a unified theory and examines the impediments and challenges to uncovering such a theory. The second book, “Logic and Mathematics”, discusses the mathematical tools commonly used in creating theories and why those tools might be inadequate for formulating a unified theory. Book III, “Modern Physics”, examines our current understanding of the Universe, embodied in classical and quantum physics.

Barring errors or misunderstandings, the discussions in the first three books employ generally accepted techniques and methods. However, the present book implements methods that radically depart from conventional approaches. This chapter explains why a venture away from commonly accepted methods is required.

1.0.1 Reductionism vs. Non-Reductionism

Presently, no single foundational system supports both the theory of relativity and quantum mechanics. The circumstance drives practitioners into adopting one of two theoretical viewpoints: 1) the ‘non-reductionist’ view, 2) the ‘reductionist’ view.

Theoretical practitioners who only require that theories predict the correct outcomes of experiments or verifiable observations are generally referred to as ‘non-reductionists’. To a non-reductionist, reconciling conflicting explanations associated with theoretical predictions is secondarily important. Embracing a plurality of theories is not viewed as disingenuous. In fact, many non-reductionists regard attempts at gaining a deeper understanding of Nature beyond correctly predicting the outcomes of experiments and observations as folly.

A reductionist, however, finds an inconsistent set of Natural laws intolerable, particularly since the evolution of physics does not suggest a plurality of laws. Historically, physics has been an exercise in unification. Newton recognized that apples fall from trees for the same reason that planets orbit the Sun. Maxwell confirmed that light, electricity and optics, once explained by independent theories, could be described by one theory. Einstein improved upon Newton’s theory by developing a relativistic field theory of
gravitation and showed that the special theory of relativity was consistent with Maxwell’s theory. The fundamental theories at the quantum level are all gauge theories, although not the same gauge theory. Reductionists maintain that there is ample evidence to suggest that the Universe can be explained by a single set of consistent laws.

1.0.2 Relativistic vs. Quantum Mathematics

As quantum mechanics evolved, practitioners recognized that the mathematics supporting general relativity was not the same as the mathematics that supported quantum mechanics, a recognition eventually proven through Bell’s theorem. The incompatibilities between general relativity and quantum physics might be overcome by discovering a “common to both” foundational system. Finding such a system is problematic, since quantum mechanics and general relativity make spectacularly successful predictions in their applicable domain. Both theories must be correct to some extent. An alternative approach must produce the same accuracy that relativity and quantum mechanics enjoy.

1.0.2.1 The Foundations of the Mathematics (Relativity and Quantum)

The theory of ‘point set topology’ is the foundation of differential geometry, the mathematical approach that supports the theory of general relativity. The fundamental concept within differential geometry is a set of independent points. Fortunately, the points can be put into one-to-one correspondence with the real numbers, allowing the construction of metric spaces in which the abstract idea of ‘distance’ can be defined numerically. The notion of ‘distance’ models extension in the physical world. Two real numbers can be added or multiplied producing a third real number, which represents a unique point. While the adding and multiplying are accomplished in sophisticated ways, the whole of general relativity involves only ordinary numbers.

The fundamental mathematical notion in quantum mechanics is the ‘state vector’. The sum total of the vectors is called a ‘vector space’. Every vector in a vector space can be written in terms of a collection of independent vectors called ‘basis vectors’. Basis vectors determine the dimension of the space. Any vector written in terms of the basis vectors is said to be in ‘superposition’. There is no such thing as a ‘superposition’ of points. All points in a topological space are independent. Adding two points (numbers) together produces another point. But adding two basis vectors together does not result in a third basis vector.

Vector spaces include scalar multiplication, where multiplying a vector by a real or complex number results in a multiple of the original vector. Moreover, quantum mechanics relies on ‘Hilbert spaces’. Hilbert spaces are vector spaces in which two vectors can be inner multiplied. Inner multiplying two vectors produces a number rather than a vector. In other words, the inner multiplication of two vectors produces a mathematical object (number) which is not a vector. When a binary operation on two objects in a mathematical space produces an object not part of the original space, the space is said to be ‘not closed’. Hilbert spaces are not closed. The fact that point set topology is mathematically closed, while Hilbert spaces are not, makes the mathematics of quantum mechanics markedly different from the mathematics of general relativity.
The challenge, then, is to find some common ground between these two distinctly different systems of logic (mathematics).

1.1 The Philosophy of Logic

Intuitively, logic, emergent in the Western World for about 2500 years or more, should be a well-substantiated field of study, with only minor theoretical issues left unresolved. But the scope of the subject has broadened over the years. Logic now includes virtually all of mathematics. Some philosophers believe that logic and mathematics are separate disciplines, but agreement on where to draw the line between the two remains elusive. It is more convenient to consider mathematics and logic as the same specialty. However, logic suffers from a myriad of philosophical issues which promote turmoil amongst its practitioners and it will be educational to discuss some of the issues.

In what follows, the question “what is logic?” is framed in terms of the philosophical perspectives of those logisticians generally referred to as ‘nominalists’ and those referred to as ‘realists’. The type of realism discussed here is commonly called ‘Platonism’. There are additional perspectives on realism and nominalism that will not be discussed, but the reader can consult alternative sources for additional viewpoints.

1.1.1 Logical Realism

Logical realism is the philosophical view that logic is independent of language, thought and practices. In general, realists believe that mathematical objects exist whether or not anyone thinks about them. To a realist, if electrons and planets exist independently of an observer, so do numbers and sets. Statements about electrons and planets, which evidently represent ‘ideas’ about electrons and planets, are true if the statements accurately reflect the properties of real electrons and planets. The same can be said of statements about numbers and sets. The logical realist, then, believes that mathematical truths are discovered as opposed to the ‘idea’ that mathematics is invented [257].

Philosophers have developed a variety of objections to logical realism, claiming that abstract mathematical objects are inaccessible, making them problematic. Logical realism has been among the most hotly debated topics in the philosophy of mathematics over the past few decades [257].

Logical realism can be characterized by:

1. **Existence**: Mathematical objects exist
2. **Abstractness**: Mathematical objects do not exist in space-time
3. **Independence**: Mathematical objects exist independently of thought
4. **Universal**: a common property in a collection of objects

‘Existence’ is the idea that mathematical objects exist independently of language, thought and practices. This view can be formalized as

$$\exists x (x \in M \equiv x \text{ is a mathematical object}),$$
which is true whether anyone thinks about the mathematical object or not.
‘Abstractness’ is the idea that mathematical objects do not exist in space and time. Abstractions are often called ‘logical forms’, which exist in an independent world of ‘Platonic forms’. ‘Independence’ is the idea that if there were not intelligent beings, or had their language, thought or practices been different, mathematical objects would still exist [251]. Finally, a ‘universal form’ is a common property that all objects in a collection possess.

Logical realism affirms universals and abstractions. A nonmathematical example of a universal is ‘cowness’. There are many cows, but only one ‘universal form’ of a cow. An example of an abstraction is a ‘number’. Numbers do not exist in space and time.

1.1.2 Logical Nominalism

Nominalism comes in at least two varieties. One variety rejects the idea of an ‘abstraction’. The other rejects the idea of a ‘universal’. The two varieties are independent and either belief can be consistently held without the other. However, both varieties share some common motivations and arguments. A universal is a kind of ‘form’, a common property possessed by distinct entities. An abstraction is neither spatial nor temporal [252]. Hypothetically, an orthodox nominalist would deny both universals and abstractions.

Nominalism is not simply the rejection of universals and abstractions. If that were the case, a nihilist, who believes in no entities at all, would count as a nominalist. The term ‘nominalism’ implies that everything is ‘particular’ or ‘concrete’, and that this is not trivially so. Thus, one kind of nominalism asserts that there are particular objects and that everything is particular and the other asserts that everything is concrete. In other words, everything that exists, exists in space and time.

Nominalism, in both senses, is a kind of anti-realism. One kind of nominalism denies the existence of universals and the other denies the existence of abstractions. Some nominalists accept the existence of properties, propositions, possible worlds and numbers, but only on account of being particular or concrete objects. And rejecting properties, propositions, possible worlds and numbers as real, and any other entities not spatial or temporal, is not sufficient for being a nominalist; a nominalist must reject them on account of being universals or abstractions [252]. Unlike the nihilist, both nominalists and realists agree that certain objects exist, but disagree on the nature of those objects. A nominalist would not deny the existence of a ‘cow’, but only that there is anything like the ‘universal form’ of a cow. A cow consists of a particular collection of sense-data, which identifies them as a particular thing, nothing more.

The nominalism discussed here rejects both universals and abstractions. The point is not to debate the merits of nominalism vs. realism, but to compare and contrast two views on the opposite ends of the logical philosophical spectrum so as to ferret out the salient points and, thereby, gain a better understanding of “what is logic?”.
1.1.3 Nominalism vs. Realism in Logic

The debate between the ‘nominalist’ and the ‘realist’ about logic revolves around questions of syntax (valid logical forms) and semantics (what logical statements actually say). Despite the disagreements, there are certain language forms that virtually all logicians agree are part of logic. For instance,

1. All $S'$s are $M'$s, all $M'$s are $P'$s, therefore all $S'$s are $P'$s
2. $x = x$
3. $p \land \sim p$
4. $p \lor \sim p$

But there is considerable disagreement regarding the interpretation of these forms. Philosophers and logicians, that regard ‘sets’, ‘classes’ and ‘numbers’ as somehow “make-believe”, are usually called ‘logical nominalists’. On the other hand, those philosophers and logicians who regard ‘sets’, “classes’ and ‘numbers’ as independent objects are called ‘logical realists’. A nominalist is not likely to say:

A. For all classes ‘$S, M, P$’, if all $S'$s are $M'$s, all $M'$s are $P'$s, then all $S'$s are $P'$s

A nominalist would more likely say:

B. The following turns into a valid form no matter what words or phrases of the appropriate kind are substituted for the letters ‘$S, M, P$’ in the sentence ‘if all $S'$s are $M'$s, all $M'$s are $P'$s, then all $S'$s are $P'$s’

The issue is that the nominalist does not believe that abstractions, such as ‘classes’, exist. To a nominalist, sentences, words and letters are concrete, existing in space and time. Hence, the preference for formulation ‘B’ over formulation ‘A’ [157].

As to the form ‘$x = x$’, some philosophers contend that the symbol ‘$=$’, representing the reflexive relation, is not a relation, arguing that relations make sense only if one object bears a relationship to a different object. They argue that whatever ‘$=$’ represents, it is not a relation [157].

As to ‘3’ and ‘4’, the issue revolves around what ‘$p$’ stands for. Nominalists are likely to say that $p$ is any appropriate sentence. Whereas realists find it patently absurd that logic has anything to do with sentences, but only with what the sentences say. To a realist, the assignment of a true/false value to $p$ falls outside logic.

A realist is likely to say that “the moon is spherical” if and only if the moon is spherical. To a realist, the sentence “the moon is spherical” is simply a representation of an ‘idea’. To be true or false, the idea must be instantiated by an external (to logic) state of affairs. The instantiation is what makes the sentence true or false rather than the sentence itself. Sentences are just “mounds of ink on paper”, the disappearance of which would not erase the idea expressed by the sentence. The nominalist contention that a sentence or a list of sentences at face value could be meaningful is rejected [157].
But the nominalist would retort that one of the objectives of logic is to find language forms that are valid independently of the sentences that comprise the form. Such a form is called a ‘tautology’. For instance, the form ‘\( p \lor \sim p \)’ is a tautology. It is always true so long as \( p \) is an appropriate sentence.

Evidently, this does not satisfy the realist. A logical ‘argument’ or ‘schema’ is a list of sentences all of which are ‘premises’ except one, the ‘conclusion’. In logic, there are valid and invalid arguments. For instance, “all \( S \)'s are \( M \)'s, all \( M \)'s are \( P \)'s, therefore all \( S \)'s are \( P \)'s” is a valid argument. But the argument “if some \( S \)'s are \( M \)'s, then all \( S \)'s are \( M \)'s” is invalid. Defining a valid argument requires the notion of ‘truth’, a concept that the nominalist would like to eliminate, since it does not represent a concrete experience. But a valid schema ‘\( S \)’ has the characteristic that if the premises in \( S \) are true, then so is the conclusion in \( S \). Invalid schemas violate this rule. While the nominalist would like to rid logic of abstractions such as ‘truth’, so far, all attempts at writing a logical language void of abstractions have failed.

1.1.4 The Problem of a Single Set System of Logic

Book II, Sec. 3.2.2.1 explained that, except for the use of different symbols, the rules associated with the ‘predicate calculus’ are identical to the rules associated with ‘Cantorian set theory’. Fundamentally, propositional logic and set theory are logically identical.

Cantor’s definition of a ‘set’: a collection of objects into a whole of our intuition or our thought. Hence, \( M = \{m\} \) refers to a set ‘\( M \)’ comprised of elements ‘\( m \)’. Cantor’s definition raises two questions: 1) what are the elements ‘\( m \)’ thoughts or ideas of? 2) in what sense are the elements ‘\( m \)’ a ‘collection’?

1.1.4.1 Ideas vs. Sentences

To the first question, the nominalist would prefer that the elements ‘\( m \)’ refer to concrete objects, perceptible through sense experience, existing in space and time. If the elements ‘\( m \)’ signify ideas, of what do those ideas consist? It is hard to see how ideas could be identical to “mounds of ink on paper”. Other than a vehicle for communication, the “mounds of ink” seem superfluous. Evidently, the nominalist argues that the human mind can identify certain configurations of “mounds of ink” as a ‘sentence’ opposed to a configuration of words and letters that is not a sentence. But an explanation of what mental capacity supports this capability is metaphysically elusive.

The realist would further retort that ideas are not concrete objects. Ideas do not exist in space and time. And it is irrelevant that ideas are represented by “mounds of ink” on paper. Those mounds of ink represent abstractions, not concrete objects. The “mounds of ink” are meaningful only if correlated to something else. This argument is not unlike Kant’s objection to Berkeley’s claim that an external world did not exist. To Kant, the human mind can associate itself with a human body, and furthermore, can distinguish between its body and bodies not its own. Kant argued that this was the only way a concept of ‘self’ could be maintained.
The realist position is that the elements ‘\(m\)’, the “mounds of ink”, must be correlated to something else. It is not the ‘sentences’ that are meaningful, but what the sentences express. And if the “mounds of ink” are related to an external state of affairs, the state of affairs is independent of the “mounds of ink” (sentences). Evidently, to the realist, this is what make ideas ‘true’ or ‘false’. The “string of words” is true or false if and only if the “string of words” can be meaningfully correlated to an external state of affairs. If the sentence, which evidently represents an idea, is instantiated by a state of affairs, the sentence is true or false. Put together meaningfully, words represent the idea of an external state of affairs, which can be represented by a relation i.e.

\[
\langle \text{string of words, state of affairs} \rangle
\]

### 1.1.4.2 The Trouble with Universals

As to the second question: upon what grounds entitles \(\{m\}\) to be thought of as a ‘collection’? Here, nominalist arguments stand on solid ground, but the realist position finds difficulties. If \(\{m\}\) signifies the idea of a ‘collection’ of \(m\) cows, what does this mean? If the intent is to say that the same name is assigned to similar objects, the nominalist likely would have little objection. But the realist wishes to say much more than this. The realist would contend that there exists something called the ‘universal form’ of a cow. And while there are many cows, there is only one ‘cow form’. But the nominalist would retort that there is nothing existing in space and time that would suggest anything like a ‘universal form’.

Where do these universal forms reside? The realist would claim that universals exists in a distinct discoverable world of forms, just as physical objects are discovered through sense perception. Every class of objects has a universal form. ‘Catness’, ‘dogness’, ‘greenness’, ‘humanness’ and ‘numberness’, all universal forms.

No doubt a nominalist would be unhappy with these assertions. The kinds of objects that the nominalist seeks, which give meaningful substance to the world emanate from sense experience, objects that exist in space and time. And there is nothing in sense experience that would suggest anything like a ‘form’ or a ‘universal’. The idea of a ‘universal form’ runs counter to everyday experience and should have no place in a conversation about the real world.

Moreover, upon what grounds can a realist say “there are \(m\) cows in the field”? Where does the idea of ‘number’ come from? There is simply nothing in the concrete world of experience that would suggest ‘number’. Again, numbers are concrete only to the extent of being marks on paper. There is nothing external to the human mind that a realist could possibly point to and say “that’s a number”. Such an object simply does not exist. Some nominalists regard numbers as concrete, but it is hard to see how “mounds of ink” could in any sense suggest ‘number’ without a supporting abstracting capability on the part of the human mind, a capability the nominalist rejects. Arguments that claim numbers are concrete seem relatively weak. See [157] for a detailed discussion of this issue.
1.2 Concluding Remarks

What does this discussion say about ‘logic’? It would seem that classes, numbers and sets will continue to be part of logic. At some point, the nominalist is compelled to accept that the human mind is capable of meaningful abstraction at some level. The direction this concession usually takes is to acknowledge that the mind has the power to abstract, but such power is limited to inventions of the human mind and does not extend to an external world. According to this view, logic and mathematics represent the ability of the mind to create, define and recognize certain language forms. Valid and invalid logical forms are but types of language forms. Such forms do not meaningfully extend to an external world. To the nominalist, logic is a set of formal linguistic rules which has nothing to do with an external world. But as to how these linguistic forms are recognized by the human mind as ‘valid’ or ‘invalid’ language forms, nominalist arguments are relatively weak.

The realist, on the other hand, demands the existence and independence of logical forms. Logical forms exist independently of thought. A realist is likely to say:

\[
a, b, \text{ and } c \text{ and so on exist, and the fact that they exist and have forms such as } F\text{-ness, } G\text{-ness, and } H\text{-ness is independent of beliefs, linguistic practices, conceptual schemes, and so on [250].}
\]

But modern logic lacks a logical form independent of a linguistic form, which is absolutely demanded in a truly realist conception of logic. The argument ‘if all \(S\)’s are \(M\)’s, all \(M\)’s are \(P\)’s, then all \(S\)’s are \(P\)’s’ seems, on its face, simply a linguistic form. Even if \(S\)’s, \(M\)’s and \(P\)’s have independent states of affairs instantiating them, where is the independent language form that represents this?

Logical realism holds that logical entities exist independently of the human mind. Thus, mathematics is not invented, but rather, discovered. This point of view suggests a sort of mathematics where discoverable objects must be represented independently of the ideas of those objects, ideas expressed through language forms.

Many working mathematicians have been or are logical realists; Gödel believed in an objective mathematical reality perceived in a manner analogous to sense perception. But while Gödel was arguably the most celebrated logician of the 20\textsuperscript{th} Century, he, like Einstein, was relegated to the sidelines, toiling in abstention while logic precipitated the highly nominalist interpretations it currently enjoys. Gödel fought this trend by showing that modern logic bore severe limitations. He showed that if a logical system was sophisticated enough to support the development of arithmetic, if it was consistent, it was incomplete. There would always be true but unprovable statements within any such system. Moreover, the axioms of the system could not be proven consistent. But Gödel did not in his lifetime produced a truly realist system of logic. This will be the objective of the remainder of this book, to answer the question: what does a truly realist system of logic entail?
Chapter 2

The Theory of Objective Logic

“A wise man is not governed by others, nor does he try to govern them; he prefers that reason alone prevail.”

~La Bruyère, Characters, 1688

2.0 Introduction

The discussion in Chapter 1 contrasted the philosophy of ‘nominalism’ with ‘realism’. The two perspectives represent the opposite ends of the philosophical spectrum as it pertains to logic.

This chapter develops a logical approach, which could be described as ‘radical realism’, called ‘objective logic’. The new approach will uphold the existence of abstract mathematical objects, the ideas of which are expressed in a ‘language’, but instantiated by an independent ‘language’. It will affirm the independent existence of ideas and the instantiations of those ideas. No attempt is made to defend the merits of the approach, but the discussion focuses on what the new approach entails.

2.1 The Foundations of Logical Systems

Systems of logic generally share a common set of attributes:

1. **The Object Language**: the syntax of the logical system
2. **Forms**: the semantics of the object language
3. **Consistent and Inconsistent**: compatible/incompatible semantics
4. **Completeness and Incompleteness**: provable/unprovable

2.2 The Syntax of Objective Logic

A set ‘$S$’, called the ‘nominal’ set, is a collection of elements ‘$s$’ into a whole ‘$S$’ of intuition or thought, designated ‘$S = \{s\}$’. A set ‘$\bar{S}$’, called the ‘instantiation set’, independent of $S$, is a collection of elements ‘$\bar{s}$’ into a whole, designated ‘$\bar{S} = \{\bar{s}\}$’. The syntax of objective logic involves relating $S$ to $\bar{S}$.

2.2.1 The Object Language

If $s \in S$, then $s$ is a name for an expression of thought. If $\bar{s} \in \bar{S}$, then $\bar{s}$ represents a name for the instantiation of the thought in $S$. Hence,

$$\forall s \forall \bar{s} (s \in S \land \bar{s} \in \bar{S} \leftrightarrow \bar{s} \notin S \land s \notin S)$$
For instance, ‘∅’ is a name that represents an expression for the thought of ‘nothing’ or ‘emptiness’; the symbol ‘⁻∅’ is a name that represents the state of ‘nothing’ or ‘emptiness’.

The relationship between S and ̅S is defined by a relation ‘r’ that associates the elements in S with the elements in ̅S, designated ‘S r ̅S’ and a relation ‘⁻r’ that associates the elements in ̅S with the elements in S, designated ‘⁻S r S’. Note that ⁻r instantiates r.

Example (see fig. 2.2-1): Let S = {s₁, s₂, s₃, s₄} and ̅S = {̅s₁, ̅s₂, ̅s₃}, then r associates S with ̅S.

Figure 2.2-1

2.2.2 Statements (the Semantics of Objective Logic)

The form ‘r(sᵢ) = ̅sⱼ’ is called a ‘statement’. If sᵢ ∈ S, ̅sⱼ ∈ ̅S, by fiat, r(sᵢ) = ̅sⱼ is a ‘true’ statement, otherwise, it is ‘false’. Statements are either true or false. For instance, in fig. 2.2-1, the statement ‘r(s₁) = ̅s₁’ is true, but ‘r(s₂) = ̅s₁’ is false.

2.2.3 Systems

A set of statements ‘S r ̅S’, where S contains n elements ‘sᵢ ∈ S, i = 1, ..., n’, and ̅S contains m elements ‘̅sᵢ ∈ ̅S, i = 1, ..., m’, is called a ‘system’. A system is a collection of statements, some of which may be true, others false. Either m or n or both can be infinite. Note that there must exist at least one statement if ‘S r ̅S’ is to be a ‘system’, denoted ‘r(sᵢ) = ̅sⱼ ∈ S r ̅S’.

2.2.3.1 Valid Systems

Evidently, the total number of possible relations ‘r’ between S and ̅S, where S contains n elements and ̅S contains m elements is 2ⁿᵐ. Hence, the total number of possible systems is 2ⁿᵐ, including the null system ‘∅ r ̅∅’. Note that r(∅) = ̅∅ is considered a statement.
The system ‘$S \xrightarrow{r} \bar{S}$’ is ‘valid’ if
\[ \forall s_i \forall \bar{s}_i (s_i \in S, \bar{s}_i \in \bar{S} \rightarrow r(s_i) = \bar{s}_i) \]

There is only one valid system within all the possible systems ‘$S \xrightarrow{r} \bar{S}$’. The necessary and sufficient conditions for validity are:

1. $r$ is one-to-one and onto
2. All the statements are true

If $r$ is one-to-one and onto, $\#S = \#\bar{S}$. The rules above hold whether $S$ is finite or infinite.

2.2.3.2 Systems of Systems

Let
\[ S = \{S_1, S_2, ..., S_n\}, \quad \bar{S} = \{\bar{S}_1, \bar{S}_2, ..., \bar{S}_m\}, \]

where $S_i, \bar{S}_i$ are sets, then $S \xrightarrow{r} \bar{S}$ is a ‘system of systems’, where $r$ now signifies a set relation. A system of systems ‘$S \xrightarrow{r} \bar{S}$’ is valid if and only if $m = n$, all the systems ‘$S_i \xrightarrow{f_i} \bar{S}_i$’ are valid and
\[ \forall s_i \forall \bar{s}_i (s_i \in S \land \bar{s}_i \in \bar{S} \rightarrow r(s_i) = \bar{s}_i) \]

2.2.3.3 Consistent and Inconsistent Systems

If $S \xrightarrow{r} \bar{S}$ is a system and
\[ \exists s_i \exists s_j (s_i, s_j \in S \rightarrow r(s_i) = r(s_j) = \bar{s}_k \in \bar{S}, \quad i \neq j), \]

then such statements are called ‘confusions’. A confusion exists if more than one distinct thought is associated with an instantiation. This is an ‘inconsistency’.

Conversely, if $S \xrightarrow{r} \bar{S}$ is a system and
\[ \exists \bar{s}_i \exists \bar{s}_j (\bar{s}_i, \bar{s}_j \in \bar{S} \rightarrow \bar{r}(\bar{s}_i) = \bar{r}(\bar{s}_j) = s_k \in S, \quad i \neq j), \]

then such statements are called ‘confounding’s’, also an ‘inconsistency’. A confounding exists if more than one distinct instantiation is associated with a thought.

Confusions and confounding’s are the only inconsistencies in objective logic. A system is consistent if $r$ is one-to-one. For suppose not, then, within an inconsistent system, either there exists
\[ r(s_i) = r(s_j) = \bar{s}_k \lor \bar{r}(\bar{s}_i) = \bar{r}(\bar{s}_j) = s_k, \quad i \neq j \]
Either case is an inconsistency. Hence, a system \( S \rightarrow \tilde{S} \) is consistent if
\[
\forall s_i \exists! \tilde{s}_j (s_i \in S, \tilde{s}_j \in \tilde{S} \rightarrow r(s_i) = \tilde{s}_j) \land \forall \tilde{s}_i \exists! s_j (s_i \in S, \tilde{s}_j \in \tilde{S} \rightarrow \tilde{r}(\tilde{s}_i) = s_j)
\]

### 2.2.3.4 Complete and Incomplete Systems

The system \( S \rightarrow \tilde{S} \) is ‘complete’ if
\[
\forall s_i \forall \tilde{s}_j \exists r (s_i \in S \land \tilde{s}_j \in \tilde{S} \rightarrow r(s_i) = \tilde{s}_j)
\]
In other words, for every element of \( S \) and every element of \( \tilde{S} \), there is at least one statement. Note that a system can be complete, but not consistent (see fig. 2.2-1).

A system \( S \rightarrow \tilde{S} \) has an ‘omission’ if
\[
\forall s_i \exists \tilde{s}_j (s_i \in S \land \tilde{s}_j \in \tilde{S} \rightarrow r(s_i) \neq \tilde{s}_j)
\]
An omission is an instantiation with no accompanying thought.

A system \( S \rightarrow \tilde{S} \) contains a ‘fabrication’ if
\[
\exists s_i \forall \tilde{s}_j (s_i \in S \land \tilde{s}_j \in \tilde{S} \rightarrow r(s_i) \neq \tilde{s}_j)
\]
A fabrication is a thought with no instantiation. Both an omission and a fabrication are ‘incomplete’ statements and are the only incomplete statements within objective logic. A system is incomplete if it contains incomplete statements. On the other hand, a system is complete if and only if it has no incomplete statements.

### 2.2.3.5 Equal and Equivalent Systems

The ‘cardinality’ of a system is designated \( \# S \rightarrow \tilde{S}_{n,m} \), signifying a system having a nominal set with \( n \) elements and an instantiation set with \( m \) elements, \( n, m \in \mathbb{N} \). Two systems \( S \rightarrow \tilde{S} \) and \( T \rightarrow \tilde{T} \) are ‘equal’ if
1. \( \# S \rightarrow \tilde{S} = \# T \rightarrow \tilde{T} \)
2. \( r = f \)
3. \( \forall s_i \forall \tilde{s}_j \forall t_i \forall \tilde{t}_j (r(s_i) = \tilde{s}_j \in S \rightarrow \tilde{s}_j \land r(t_i) = \tilde{t}_j \in T \rightarrow \tilde{t}_j \rightarrow s_i = t_i \land \tilde{s}_j = \tilde{t}_j) \)

Two systems \( S \rightarrow \tilde{S} \) and \( T \rightarrow \tilde{T} \) are ‘equivalent’ if
1. \( \# S \rightarrow \tilde{S} = \# T \rightarrow \tilde{T} \)
2. \( r = f \)

In other words, two systems are ‘equal’ if they have the same cardinality, the same syntax and the same statements. Two systems are ‘equivalent’ if they have the same cardinality and the same syntax. For example, all valid systems are equivalent.
2.3 Valid, Invalid and Partially Valid Complete and Consistent Systems

A system \( S \overset{r}{\rightarrow} \bar{S} \) is valid if and only if it is consistent and complete. In other words, \( r \) must be one-to-one and onto. However, consistency and completeness are necessary, but not sufficient conditions for validity.

2.3.1 Valid Systems

Suppose \( S \overset{r}{\rightarrow} \bar{S} \) is complete and consistent. In other words, suppose \( r \) is one-to-one and onto, then the system is ‘valid’ if and only if

\[
\forall s_i \forall \bar{s}_i (s_i \in S, \bar{s}_i \in \bar{S} \leftrightarrow r(s_i) = \bar{s}_i)
\]

In other words, the system is valid if and only if all its statements are true.

2.3.2 Complete and Consistent Invalid Systems

A complete and consistent system is ‘invalid’ if

\[
\forall s_i \forall \bar{s}_i (s_i \in S, \bar{s}_i \in \bar{S} \rightarrow r(s_i) \neq \bar{s}_i)
\]

In other words, the system is invalid if and only if all its statements are false.

2.3.3 Complete and Consistent Partially Valid Systems

A complete and consistent system is ‘partially valid’ if

\[
\exists s_i \exists \bar{s}_i (s_i \in S, \bar{s}_i \in \bar{S} \rightarrow r(s_i) = \bar{s}_i)
\]

Note that a valid system is a partially valid system whose statements are all true.

2.3.4 Proposition on Complete and Consistent Systems

Proposition 1: If \( |S| = n \), there are \( n! \) possible complete and consistent systems ‘\( S \overset{r}{\rightarrow} \bar{S} \)’, one of which is valid, \((n - 1)! \) are invalid and \( n! - (n - 1)! - 1 \) are partially valid.

Proof: There are a total of \( n! \) possible complete and consistent systems, since \( |S| = n \). There can be only one valid system, since \( r(s_i) = \bar{s}_i \) is a unique collection of statements. There are \((n - 1)\) choices for \( r(s_1) \neq \bar{s}_1 \) and \((n - 2)\) choices for \( r(s_1) \neq \bar{s}_1 \) and \( r(s_2) \neq \bar{s}_2 \). Continuing, there are \((n - 1)!\) possible invalid systems. Of the remaining possibilities, some statements are true, others false. Hence, the number of partially valid systems is \( n! - (n - 1)! - 1 \). This completes the proof.

Example: Let \( |S| = 3 \) and let

\[
S = \{s_1, s_2, s_3\}, \quad \bar{S} = \{\bar{s}_1, \bar{s}_2, \bar{s}_3\},
\]

then there are \( 3! = 6 \) possible complete and consistent systems ‘\( S \overset{r}{\rightarrow} \bar{S} \)’; note that the system
is valid. There are two possible invalid systems:

\[ r(s_1) = \bar{s}_2, \quad r(s_2) = \bar{s}_3, \quad r(s_3) = \bar{s}_1 \]

\[ r(s_1) = \bar{s}_3, \quad r(s_2) = \bar{s}_1, \quad r(s_3) = \bar{s}_2, \]

since \((3 - 1)! = 2! = 2\). There are three possible partially valid systems:

\[ r(s_1) = \bar{s}_1, \quad r(s_2) = \bar{s}_3, \quad r(s_3) = \bar{s}_2, \]

\[ r(s_2) = \bar{s}_2, \quad r(s_1) = \bar{s}_3, \quad r(s_3) = \bar{s}_1, \]

\[ r(s_3) = \bar{s}_3, \quad r(s_2) = \bar{s}_1, \quad r(s_1) = \bar{s}_2, \]

since \(6 - 2 - 1 = 3\).

### 2.4 Characteristics of a System

If \(S \xrightarrow{r} \bar{S}\) is a system, let \(T \subset S\) and \(\bar{T} \subset \bar{S}\), then \(T \xrightarrow{f} \bar{T}\) can have different characteristics from the original system \(S \xrightarrow{r} \bar{S}\), since \(f \neq r\).

#### 2.4.1 Subsystems

By insisting that \(\bar{T}\) consists of all the elements of \(\bar{S}\) that \(r\) maps to \(\bar{S}\) from \(T\), then the following applies:

**Definition:** If \(T \subset S\) and \(S \xrightarrow{r} \bar{S}\) is a system, then \(T \xrightarrow{r} \bar{T}\) is called a ‘subsystem’ of \(S \xrightarrow{r} \bar{S}\).

Let

\[ S = \{s_i\}, \quad \bar{S} = \{\bar{s}_j\}, \quad i = 1, ..., n, \quad j = 1, ..., m \]

If \(S \xrightarrow{r} \bar{S}\) is a system and

\[ T = \{s_i, s_j, s_k, ...\}, \quad s_i, s_j, s_k, ... \in S \]

such that

\[ r(s_i) = \bar{s}_a, \quad r(s_j) = \bar{s}_b, \quad r(s_k) = \bar{s}_c, ... \rightarrow \bar{T} = \{\bar{s}_a, \bar{s}_b, \bar{s}_c, ...\}, \quad \bar{s}_a, \bar{s}_b, \bar{s}_c, ... \in \bar{S} \]

then the statements

\[ r(s_i) = \bar{s}_a, \quad r(s_j) = \bar{s}_b, \quad r(s_k) = \bar{s}_c, ... \]
are exactly the same as in $S \overset{r}{\rightarrow} \bar{S}$, except restricted to $T$ and $\bar{T}$. Hence, $T \overset{r}{\rightarrow} \bar{T}$ is a 'subsystem' of $S \overset{r}{\rightarrow} \bar{S}$ denoted

$$T \overset{r}{\rightarrow} \bar{T} \subset S \overset{r}{\rightarrow} \bar{S}$$

Example: In fig. 2.2-1, $S = \{s_1, s_2, s_3, s_4\}$ and $\bar{S} = \{\bar{s}_1, \bar{s}_2, \bar{s}_3\}$. Let $T = \{s_1, s_2\} \subset S$, then $\bar{T} = \{\bar{s}_1\} \subset \bar{S}$, since $r(s_1) = \bar{s}_1$ and $r(s_2) = \bar{s}_1$. But these are exactly the same statements belonging $S \overset{r}{\rightarrow} \bar{S}$, but restricted to $T$ and $\bar{T}$. Hence,

$$T \overset{r}{\rightarrow} \bar{T} \subset S \overset{r}{\rightarrow} \bar{S}$$

2.4.2 Propositions on Systems

Proposition 2: If $T = S$ and $S \overset{r}{\rightarrow} \bar{S}$ is a system, then $T = \bar{S}$. The proof is trivial.

Proposition 3: If $S \overset{r}{\rightarrow} \bar{S}$ is a valid system and $S = \emptyset$, then $\bar{S} = \emptyset$ and $\emptyset \overset{r}{\rightarrow} \emptyset$ is a valid system.

Proof: For suppose not, then

$$\forall \bar{s} \forall S (s \in S \land \bar{s} \in \bar{S} \rightarrow r(s) = \bar{s}),$$

since $S \overset{r}{\rightarrow} \bar{S}$ is valid. But there is no element 's ∈ S', and therefore, no element '\bar{s} ∈ \bar{S}'. Since the cardinality of $S$ and $\bar{S}$ must be equal, $S = \emptyset \rightarrow \bar{S} = \emptyset$, which implies

$$\emptyset \overset{r}{\rightarrow} \emptyset$$

is a valid system. This proves the theorem.

Proposition 4: If $S \overset{r}{\rightarrow} \bar{S}$ is a system and

$$\exists s_i (s_i \in S \rightarrow r(s_i) = \bar{s}_i),$$

then

$$\exists T \exists \bar{T} (T \subset S \land \bar{T} \subset \bar{S} \rightarrow T \overset{r}{\rightarrow} \bar{T} \text{ is a valid subsystem of } S \overset{r}{\rightarrow} \bar{S})$$

Proof: Let $T$ consist of only those elements 's$_i$' such that $r(s_i) = \bar{s}_i$ for all $i$. If $T = \{s_i\}$ and, since $r(s_i) = \bar{s}_i$ for all $i$, then $\bar{T} = \{\bar{s}_i\} \subset \bar{S}$ and $T \overset{r}{\rightarrow} \bar{T}$ is valid. This completes the proof. In other words, every system with at least one true statement has a non-empty valid subsystem.
Proposition 5: If \( S \to \bar{S} \) is a complete and consistent invalid system, then
\[
\exists T \exists \bar{T} \left( T \subset S \land \bar{T} \subset \bar{S} \land T = \emptyset \land \bar{T} = \bar{\emptyset} \to T \to \bar{T} \text{ is a valid subsystem of } S \to \bar{S} \right)
\]

Proof: Since
\[
\forall s_i (s_i \in S \to r(s_i) \neq \bar{s}_i),
\]
let \( T = \emptyset \subset S \) and \( \bar{T} = \bar{T} \subset \bar{S} \). Therefore, by a previous proof, \( T \to \bar{T} \) is valid. In other words, the only valid subsystem of a complete and consistent invalid system is the null system \( \emptyset \to \bar{\emptyset} \).

2.4.3 Universal Systems

Definition: The system \( U \to \bar{U} \) is called ‘universal’ if
\[
\forall S \forall \bar{S} \left( S \to \bar{S} \subset U \to \bar{U} \right)
\]
In other words, all the systems \( S \to \bar{S} \) are subsystems of \( U \to \bar{U} \).

Proposition 6: If \( U \to \bar{U} \) is a valid system and \( S \to \bar{S} \subset U \to \bar{U} \), then \( S \to \bar{S} \) is valid.

Proof: If \( U \to \bar{U} \) is valid, then \( r \) is one-to-one and onto and all its statements are true. But if \( S \to \bar{S} \subset U \to \bar{U} \), then the statements in \( S \to \bar{S} \) are identical to the statements in \( U \to \bar{U} \), but restricted to \( S \) and \( \bar{S} \). Hence, the statements within \( S \to \bar{S} \) are one-to-one and onto and true. Therefore, \( S \to \bar{S} \) is valid. Since \( S \to \bar{S} \) is arbitrary, this completes the proof. In other words, all subsystems of valid systems are valid.

2.5 Set Operations on Valid Systems

Proposition 7: If \( S \to \bar{S} \) and \( R \to \bar{R} \) are valid systems, then the systems \( R \cap S \to \bar{R} \cap \bar{S} \) and \( R \cup S \to \bar{R} \cup \bar{S} \) are valid.

Proof: There are two cases for each claim: 1) \( R \cap S = \emptyset \), 2) \( R \cap S \neq \emptyset \). If \( R \cap S = \emptyset \), then
\[
R \cap S \to \bar{R} \cap \bar{S} = \emptyset \to \bar{\emptyset},
\]
which is valid. If \( R \cap S \neq \emptyset \), then \( \exists s (s \in S, s \in R) \). But then \( r(s) = \bar{s} \) and \( f(s) = \bar{s} \), since both \( S \to \bar{S} \) and \( R \to \bar{R} \) are valid. Hence, \( \bar{s} \in \bar{R} \cap \bar{S} \). Moreover, \( g(s) = \bar{s} \). Since \( s \) is arbitrary, \( R \cap S \to \bar{R} \cap \bar{S} \) is valid.
If \( R \cap S = \emptyset \), then \( R = \{ r_i \} \) and \( S = \{ s_i \} \), where
\[
\forall i (i \in N \Rightarrow r_i \neq s_i), \quad r(s_i) = \bar{s}_i, \quad f(r_i) = \bar{r}_i
\]
for all \( i \), since both \( S \xrightarrow{r} \bar{S} \) and \( R \xrightarrow{f} \bar{R} \) are valid. Hence,
\[
R \cup S \xrightarrow{g} \bar{R} \cup \bar{S} \rightarrow g(s_i) = \bar{s}_i, \quad g(r_i) = \bar{r}_i
\]
Therefore, \( R \cup S \xrightarrow{g} \bar{R} \cup \bar{S} \) is valid. If \( R \cap S \neq \emptyset \), then \( \exists s (s \in S, s \in R) \), \( s \) being a common element between \( R \) and \( S \). In any case, \( g(s) = \bar{s} \). Since \( s \) is arbitrary, \( R \cup S \xrightarrow{g} \bar{R} \cup \bar{S} \) is valid. This completes the proof.

Example: Let
\[
R = \{ a, b \}, \quad \bar{R} = \{ \bar{a}, \bar{b} \}, \quad S = \{ a, c, d \}, \quad \bar{S} = \{ \bar{a}, \bar{c}, \bar{d} \}
\]
Let \( R \xrightarrow{r} \bar{R} \) and \( S \xrightarrow{f} \bar{S} \) be valid systems. Hence,
\[
r(a) = \bar{a}, \quad r(b) = \bar{b}, \quad f(a) = \bar{a}, \quad f(c) = \bar{c}, \quad f(d) = \bar{d}
\]
Now \( R \cap S = \{ a \} \) and \( \bar{R} \cap \bar{S} = \{ \bar{a} \} \). But \( r(a) = \bar{a} \) and \( f(a) = \bar{a} \). Hence, \( R \cap S \xrightarrow{g} \bar{R} \cap \bar{S} \) is valid, since \( g(a) = \bar{a} \). Now,
\[
R \cup S = \{ a, b, c, d \}, \quad \bar{R} \cup \bar{S} = \{ \bar{a}, \bar{b}, \bar{c}, \bar{d} \}
\]
But
\[
r(a) = \bar{a}, \quad r(b) = \bar{b}, \quad f(c) = \bar{c}, \quad f(d) = \bar{d}
\]
Hence, \( R \cup S \xrightarrow{g} \bar{R} \cup \bar{S} \) is valid, since \( g(a) = \bar{a} \) and so on.

2.6 Functions on Systems

Proposition 8: Suppose \( A \xrightarrow{r} \bar{A} \) is valid, let there exist a system \( 'B \xrightarrow{f} \bar{B}' \). If there is a function \('g'\) such that \( A \xrightarrow{g} B \) and a function \('\bar{g}'\) such that
\[
\forall a (a \in A \land g(a) = b \rightarrow \bar{g}(\bar{a}) = \bar{b})
\]
then \( (B \xrightarrow{g_c} A) \xrightarrow{t} (\bar{A} \xrightarrow{\bar{g}} \bar{B}) \) is valid.

Proof: Suppose \( g(a) = b, \ a \in A, \ b \in B \), then \( g_c(b) = a \), where \( g_c \) is the converse of \( g \). Since \( A \xrightarrow{r} \bar{A} \) is valid, \( \forall a (a \in A \rightarrow r(a) = \bar{a}) \), and furthermore, \( \bar{g}(\bar{a}) = \bar{b} \). Therefore,
\(\forall b\,(t(b) = \bar{b}).\) Since \(b\) is arbitrary, \((B \xrightarrow{g \circ c} A) \xrightarrow{t} \bar{A} \xrightarrow{\bar{g}} \bar{B}\) is valid. This completes the proof. Note that \(g\) is instantiated by \(\bar{g}\).

### 2.6.1 Composite Functions on Systems

**Proposition 9:** Suppose \(A \xrightarrow{r} \bar{A}\) is valid and let there exist systems \(\bar{B} \xrightarrow{s} B\) and \(C \xrightarrow{t} \bar{C}\).

Let \((A \xrightarrow{f} B \xrightarrow{g} C)\), if there exist functions

\[f(a) = b \land g(b) = c \Rightarrow \bar{f}(\bar{a}) = \bar{b} \land \bar{g}(\bar{b}) = \bar{c}, \quad a \in A, \quad b \in B, \quad c \in C\]

then \((C \xrightarrow{(g \circ f)} A) \xrightarrow{h} (\bar{A} \xrightarrow{\bar{g} \circ \bar{f}} \bar{C})\) is valid.

Proof: \(f(a) = b \land g(b) = c \Rightarrow \bar{f}(\bar{a}) = \bar{b} \land \bar{g}(\bar{b}) = \bar{c}\), then

\[(g \circ f)(a) = c \Rightarrow (g \circ f)_c(c) = a\]

But since \(A \xrightarrow{r} \bar{A}\) is valid \(\forall a\,(r(a) = \bar{a})\) and \((\bar{g} \circ \bar{f})(\bar{a}) = \bar{c}\). Hence, \(h(c) = \bar{c}\). Since \(c\) is arbitrary, \((C \xrightarrow{(g \circ f)} A) \xrightarrow{h} (\bar{A} \xrightarrow{\bar{g} \circ \bar{f}} \bar{C})\) is valid. This proves the theorem.

### 2.6.2 Valid Functions

**Proposition 10:** Suppose \(A \xrightarrow{r} \bar{A}\) is valid and \(B \xrightarrow{s} \bar{B}\) is a system. If \(\{f, g, h, \ldots\}\) and \(\{\bar{f}, \bar{g}, \bar{h}, \ldots\}\) are sets of functions from \(A\) into \(B\) and from \(\bar{A}\) into \(\bar{B}\) respectively, then

\[
\{f, g, h, \ldots\} \xrightarrow{t} \{\bar{f}, \bar{g}, \bar{h}, \ldots\}
\]

is valid if and only if \(t(f) = \bar{f}, t(g) = \bar{g}, t(h) = \bar{h}, \ldots\), where

\[
\forall a \forall b\,(a \in A, b \in B \land f(a) = b \Rightarrow \bar{f}(\bar{a}) = \bar{b})
\]

Proof: Since \(A \xrightarrow{r} \bar{A}\) is valid,

\[\forall a\,(a \in A \rightarrow r(a) = \bar{a})\]

If \(f(a) = b\), then \(f_c(b) = a\) and \(r(a) = \bar{a}\). In order for \((B \xrightarrow{f_c} A) \xrightarrow{t} (\bar{A} \xrightarrow{\bar{f}} \bar{B})\) to be valid, \(\bar{f}(\bar{a}) = \bar{b}\). Since \(a, b\) and \(f\) are arbitrary,

\[
\forall a \forall b \forall f\,(B \xrightarrow{f_c} A) \xrightarrow{t} (\bar{A} \xrightarrow{\bar{f}} \bar{B}) \text{ is valid} \Rightarrow f \xrightarrow{t} \bar{f}
\]

This completes the proof.
Moreover, if \( A \overset{r}{\rightarrow} A \) is valid, then \( r \) is one-to-one and onto, and hence, has an inverse \( \tilde{r}^{-1} \) such that \( r(a) = \tilde{a} \leftrightarrow \tilde{r}^{-1}(\tilde{a}) = a \). Hence,

\[
(\tilde{r}^{-1} \circ r)(a) = a, \quad (r \circ \tilde{r}^{-1})(\tilde{a}) = \tilde{a}
\]

Therefore,

\[
\forall a \forall \tilde{a} ( (\tilde{r}^{-1} \circ r)(a) \overset{f}{\rightarrow} (r \circ \tilde{r}^{-1})(\tilde{a}) )
\]

is valid.

### 2.7 Relations

If \( A \overset{r}{\rightarrow} A \) and \( B \overset{f}{\rightarrow} B \) are systems, then

\[
A \times B \overset{g}{\rightarrow} \tilde{A} \times \tilde{B}
\]

is a system, where

\[
\langle x, y \rangle \in A \times B, \quad x \in A, \quad y \in B, \quad \langle \tilde{x}, \tilde{y} \rangle \in \tilde{A} \times \tilde{B}, \quad \tilde{x} \in \tilde{A}, \quad \tilde{y} \in \tilde{B}
\]

The ordered pair \( \langle x, y \rangle \in A \times B' \) is called a ‘relation’ in \( A \) and \( B \). If

\[
\exists x \exists y ( (x, y) \in A \times B \rightarrow g[ (x, y) ] = (\tilde{x}, \tilde{y}) \in \tilde{A} \times \tilde{B} ),
\]

then the relation is ‘true’, otherwise, it is ‘false’.

#### 2.7.1 Valid Relations

Given a system of relations ‘\( A \times B \overset{g}{\rightarrow} \tilde{A} \times \tilde{B} \)’, let \( C' \subset A \times B \) such that

\[
\forall a \forall b ( (a, b) \in C' \subset A \times B \land (\tilde{a}, \tilde{b}) \in \tilde{C'} \rightarrow g((a, b)) = (\tilde{a}, \tilde{b}) \in \tilde{C'} ),
\]

then \( C' \overset{g}{\rightarrow} \tilde{C}' \) is valid. In other words, all the relations belonging to ‘\( C' \overset{g}{\rightarrow} \tilde{C}' \)’ are true and \( g \) is one-to-one and onto. For example, if

\[
\forall a ( (a, a) \in C' \subset A \times A \rightarrow g((a, a)) = (\tilde{a}, \tilde{a}) \in \tilde{C'} ),
\]

then the relation is called ‘reflexive’. If

\[
\forall a \exists! b ( (a, b) \in C' \subset A \times B \rightarrow g((a, b)) = (\tilde{a}, \tilde{b}) \in \tilde{C'} ),
\]

then the relation is called a ‘function’.
2.7.2 Ordering Relations

Order in mathematics is a relation \( R = (A, A, C) \), where if \( a, b \in A \), then

\[
(a, b) \in A \times A \rightarrow a \preceq b \in C
\]

(reads ‘\( a \) precedes \( b \)’). Hence, \( a \preceq b \) induces an order on the elements belonging to \( A \).

2.7.2.1 Partial and Total Orderings

If there exists a relation \( R = (A, A, C) \) such that

1. \( \forall a(a \in A \rightarrow a \preceq a \in C) \),
2. \( \forall a \forall b(a, b \in A \land a \preceq b \in C \land b \preceq a \in C \rightarrow a = b) \),
3. \( \forall a \forall b \forall c(a, b, c \in A \land a \preceq b \in C \land b \preceq c \in C \rightarrow a \preceq c \in C) \),

then \( R \) is called a ‘partial’ ordering of \( A \). Note that if \( A \) is partially ordered by \( R \), then \( A \) is also partially ordered by \( R^{-1} \), where the precedence is reversed i.e. \( a \succeq b \). If \( a \preceq b \in C \) and \( a \preceq c \in C \), but \( b \preceq c \in C \), then \( b \) and \( c \) are ‘not comparable’ signified ‘\( b \not\sim c \)’. A partial ordering, where all the elements of \( A \) are comparable, is called a ‘total’ ordering of \( A \).

Suppose \( B \subset A \), then \( R \) induces the same partial ordering in \( B \) as it does in \( A \), but restricted to \( B \) i.e. \( R' = (B, B, C) \). In general, all the subsets of \( A \) will have the same ordering as \( A \), restricted to that subset. Note that some subsets of \( A \) may be totally ordered by \( R \). If \( A \) is totally ordered by \( R \), then all subsets of \( A \) will be totally ordered.

2.7.2.2 First, Last, Maximal and Minimal Elements

If \( \forall x \exists! a(a, x \in A \rightarrow a \preceq x \in C) \), then \( a \) is called a ‘first element’ in \( R = (A, A, C) \).

Conversely, if \( \forall x \exists! a(a, x \in A \rightarrow a \succeq x \in C) \), then \( a \) is called a ‘last element’ in \( R = (A, A, C) \).

If \( \forall x \exists a(a, x \in A \rightarrow a \preceq x \in C) \), then \( a \) is called a ‘minimal element’ in \( R = (A, A, C) \).

If \( \forall x \exists a(a, x \in A \rightarrow a \succeq x \in C) \), then \( a \) is called a ‘maximal element’ in \( R = (A, A, C) \).

If \( a \in A \) is a first element, then \( a \) is a minimal element in \( R = (A, A, C) \) and is unique. If \( a \in A \) is a last element, then \( a \) is a maximal element in \( R = (A, A, C) \) and is unique.

If \( R = (A, A, C) \) is a partial ordering of \( A \), \( R \) may have many maximal and/or minimal elements. If \( a \) and \( b \) are both minimums or both maximums in \( R \), then \( a \not\sim b \) in \( R \).

If \( R = (A, A, C) \) is a total ordering on \( A \), then \( R \) can have at most one maximal and one minimal element. Every finite partially ordered set has at least one maximal and at least one minimal element. An infinitely ordered set need not have a maximal or minimal element [179].
2.7.2.3 Upper and Lower Bounds

Let $B \subset A$, which is partially ordered. If

$$\forall x \exists m (x \in B \land m \in A \rightarrow m \preceq x),$$

then $m$ is called a ‘lower bound’ of $B$. If

$$\forall x \exists m (x \in B \land m \in A \rightarrow m \succeq x),$$

then $m$ is called an ‘upper bound’ of $B$.

If $M$ is a lower bound of $B$ and if

$$\forall m \forall x \exists M (x \in B \land m \in A \land m \preceq x \rightarrow M > m),$$

then $M$ is called the ‘greatest lower bound’ of $B$ or ‘$\text{inf}(B)$’.

If $M$ is an upper bound of $B$ and if

$$\forall m \forall x \exists M (x \in B \land m \in A \land m \succeq x \rightarrow M < m),$$

then $M$ is called the ‘least upper bound’ of $B$ or ‘$\text{sup}(B)$’. There can be at most one $\text{inf}(B)$ and one $\text{sup}(B)$.

2.7.2.4 Similarity of Two Ordered Sets

Suppose there are two sets ‘$A, B$’ which are ordered by the ordering relations $R$ and $R'$ respectfully. If $A \xrightarrow{f} B$, where $f$ is one-to-one and onto such that

$$\forall a \forall a' (a, a' \in A \land f(a), f(a') \in B \land a \preceq a' \leftrightarrow f(a) \preceq f(a')),$$

then $f$ is called a ‘similarity mapping’ from $A$ into $B$ and $A$ and $B$ are said to be ‘similar’, designated ‘$A \simeq B$’. If $R$ totally orders $A$ and $A \simeq B$, then $R'$ totally orders $B$ in the same order as $R$ orders $A$. If $a \in A$ is a first (last, minimal, maximal) element and $A \simeq B$, then $f(a) \in B$ is a first (last, minimal, maximal) element in $B$. If $A \simeq B$, then $\#A = \#B$.

2.8 Validly Ordered Systems

In objective logic, order must be instantiated.

Proposition 11: Let $A \xrightarrow{f} \bar{A}$ be valid and let $R = (A, A, C)$ signify a relation on $A$ and $\bar{R} = (\bar{A}, \bar{A}, \bar{C})$ signify a relation on $\bar{A}$. If $C$ is an ordering relation i.e.

$$\forall a \forall b ((a, b) \in C \rightarrow a \preceq b),$$

then $R \xrightarrow{f} \bar{R}$ is valid if and only if $A \simeq \bar{A}$. 
Proof: If $A \simeq \bar{A}$, then

\[ \forall a \forall b \left(a \leq b \in R \rightarrow \bar{a} \leq \bar{b} \in \bar{R}\right) \]

But then there exists an ‘$f$’ such that

\[ \forall a \forall b \left(\langle a, b \rangle \in C \rightarrow f[a \leq b] = \bar{a} \leq \bar{b} \in \bar{C}\right) \]

and so $R \rightarrow \bar{R}$ is valid, where $f$ is one-to-one and onto. If $R \rightarrow \bar{R}$ is valid, then $f[a \leq b] = \bar{a} \leq \bar{b}$ for all $a \leq b \in C$. But then, $A \simeq \bar{A}$. This completes the proof.

If $R \rightarrow \bar{R}$ is valid and $R$ induces a total ordering on $A$, then $\bar{R}$ is totally ordered. This should be plain, since, if $R \rightarrow \bar{R}$ is valid, then $A \simeq \bar{A}$ and $\bar{R}$ must be a total ordering. In general, if $R \rightarrow \bar{R}$ is valid, then $\bar{R}$ is simply an instantiation of the ordering ‘$R$’ on $A$. Hence, if $a \in A$ is a first element in $R = (A, A, C)$, then $\bar{a} \in \bar{A}$ is a first element in $\bar{R} = (\bar{A}, \bar{A}, \bar{C})$. If $a \in A$ is a maximal element in $R = (A, A, C)$, then $\bar{a} \in \bar{A}$ is a maximal element in $\bar{R} = (\bar{A}, \bar{A}, \bar{C})$ and so on.

In fact, a stronger theorem can be provided. It is not necessary that $A \rightarrow \bar{A}$ be valid. It is only necessary that

\[ \forall a \forall b \left(f(a) = \bar{a}' \land f(b) = \bar{b}' \land a \leq b \in R \rightarrow \bar{a}' \leq \bar{b}' \in \bar{R}\right) \]

In other words, invalid systems can be validly ordered.

Proposition 12: If two systems ‘$(A, A, C) \rightarrow^h (\bar{A}, \bar{A}, \bar{C})$, $(B, B, C) \rightarrow^g (\bar{B}, \bar{B}, \bar{C})$’ are valid orderings and if $A \simeq B$, then $(A, A, C) \rightarrow (\bar{B}, \bar{B}, \bar{C})$ is a valid ordering.

Proof: Since $A \simeq B$, then

\[ \forall a \forall a' \exists f \left(a \leq a' \in (A, A, C) \rightarrow f(a) \leq f(a') \in (B, B, C)\right) \]

But since

\[ (A, A, C) \rightarrow^h (\bar{A}, \bar{A}, \bar{C}), \quad (B, B, C) \rightarrow^g (\bar{B}, \bar{B}, \bar{C}) \]

are valid orderings, then

\[ \forall a \forall b \left(a, b \in A \land a \leq b \in (A, A, C) \land h(a \leq b) = \bar{a} \leq \bar{b} \in (\bar{A}, \bar{A}, \bar{C}) \land g(f(a) \leq f(b)) \right. \]

\[ \left. \in (B, B, C) = f(\bar{a}) \leq f(\bar{b}) \in (\bar{B}, \bar{B}, \bar{C}) \rightarrow a \leq b \in (A, A, C) \rightarrow f(\bar{a}) \leq f(\bar{b}) \in (\bar{B}, \bar{B}, \bar{C}) \right) \]

Therefore, $(A, A, C) \rightarrow (\bar{B}, \bar{B}, \bar{C})$ is a valid ordering. This completes the proof.
2.8.1 Well-Ordered Systems

If \( R \) totally orders \( A \) and \( a \in A \) is a first element, then \( A \) is said to be ‘well-ordered’ by \( R \). Let \( (A, A, C) \) be an ordering on the set ‘\( A \)’. If

\[
\forall A' \forall x \exists a (A' \subset A \land a, x \in A' \rightarrow a < x \in (A', A', C)),
\]

then all the subsets of \( (A, A, C) \) have a first element. In this case, \( (A, A, C) \) is called a ‘well-ordered’ set. If \( (A, A, C) \) is valid and \( (A, A, C) \) is a well-ordered set, then \( (A, A, C) \) is called a ‘well-ordered system’. Every subsystem of \( (A, A, C) \) is validly well-ordered and if \( A = B, (A, A, C) \) is validly well-ordered.

Let \( A = \{a_1, a_2, ..., a_n\} \) and let \( A' = \{a_{i_1}, a_{i_2}, ..., a_{i_n}\} \) be the set ‘\( A' \)’ arranged according to order. Hence, \( A' \) is a well-ordered set. If \( B' = \{b_{i_1}, b_{i_2}, ..., b_{i_n}\} \) is well-ordered and \( A' \cap B' = \emptyset \), then \( A' \cup B' = \{a_{i_1}, a_{i_2}, ..., a_{i_n}; b_{i_1}, b_{i_2}, ..., b_{i_n}\} \) is well-ordered. In general, all finite well-ordered sets with the same number of elements are similar to one another.

2.8.2 Transfinite Induction

If \( S \subset A \) and \( A \) is well-ordered such that

1. \( a_0 \in S \),
2. \( s(a) \subset S \rightarrow a \in S \),

where \( a_0 \in A \) is the first element of \( A \) and \( s(a) \) is called the ‘initial segment’ of \( a \) such that \( s(a) < a \), then \( S = A \).

Proof: Suppose \( S \neq A \), then \( A \sim S = T \neq \emptyset \). Since \( A \) is well-ordered, \( T \) has a first element ‘\( t_0 \)’. If \( x \in s(t_0) \), then \( x < t_0 \). Hence, \( x \notin T \), but \( x \in S \), since \( s(t_0) \subset S \). But by 2, \( t_0 \in S \). But this contradicts the fact that \( t_0 \in A \sim S \). Hence, the original assumption that \( S \neq A \) leads to a contradiction. In other words, \( S = A [179] \). This proves the theorem.

If \( A \) is a well-ordered set, then

\[
\forall x \exists a (a \in A \land x \in s(a) \rightarrow x < a)
\]

Proposition 13: If \( S(A) \) symbolizes the family of all initial segments of elements in a well-ordered set ‘\( A \)’ and \( S(A) \) is ordered by set inclusion, then \( A \sim S(A) \). In particular, if \( A \) is defined by

\[
\forall x \exists f (x \in A \rightarrow f(x) = s(x)),
\]

then \( f \) is a similarity mapping of \( A \) into \( S(A) \).

Proof: By definition, \( f \) is onto. Suppose \( x \neq y \), then either \( x < y \) or \( y < x \). If \( x < y \), then \( x \notin s(y) \), but \( y \notin s(x) \). Hence, \( s(y) \neq s(x) \) and \( f \) is one-to-one. It is easily shown that \( f \)
preserves order i.e. \( x \leq y \rightarrow s(x) \subset s(y) \). Hence, \( f \) is a similarity mapping of \( A \) into \( S(A) \). This completes the proof.

If \( A \xrightarrow{f} \bar{A} \) is valid and \( A \) is a well-ordered set, then if \( f(a_0) = \bar{a}_0, \) \( a_0 \) being the first element in \( A \) and \( \bar{a}_0 \) the first element in \( \bar{A} \) and if \( s(a) \subset S \rightarrow f[s(a)] = \bar{s}(\bar{a}) \), \( \bar{a} \in \bar{S}, \) then \( A \sim \bar{A} \) and \( A \xrightarrow{f} \bar{A} \) is validly well-ordered.

2.8.2.1 Limit Points in a Well-Ordered Set

If

\[
\forall a \forall b \exists c(a, b \in A, a < b \rightarrow a < c < b),
\]

then \( a \) is the ‘immediate predecessor’ of \( b \) and \( b \) is the ‘immediate successor’ of \( a \). Every element of a well-ordered set has an immediate successor except the last element. There is no analogous statement about immediate predecessors. An element in a well-ordered set is called a ‘limit point’ if it does not have an immediate predecessor and if it is not the first element [179].

Example: Let \( R_q \) be the set of ‘rational’ numbers. No element of \( R_q \) has an immediate predecessor or an immediate successor. To see this,

\[
\forall a \forall b \left( a, b \in R_q \land a < b \rightarrow a < \frac{a + b}{2} < b \in R_q \right)
\]

Hence, \( R_q \) is not a well-ordered set, but all the elements of \( R_q \) are ‘limit points’.

Example: Let \( D = \{1,3,5,\ldots\} \) and \( E = \{2,4,6,\ldots\} \), then \( \langle D; E \rangle = \{1,3,5,\ldots; 2,4,6,\ldots\} \); the pair ‘\( \langle D; E \rangle \)’ signifies the set ‘\( D \cup E \)’ ordered position-wise from left to right i.e. any element in \( D \) precedes any element in \( E \), and elements in the same set keep the same order [179]. Since the element ‘2’ in \( \langle D; E \rangle \) has no immediate predecessor, it is a ‘limit point’.
Moreover, the element ‘2’ is the only limit point in \( \langle D; E \rangle \).

2.8.2.2 Ordinal Numbers

If a well-ordered set ‘\( A \)’ is equal to an initial segment of a well-ordered set ‘\( B \)’, then \( A \) is ‘shorter’ than \( B \). Hence, for any two well-ordered sets ‘\( A, B \)’, either 1) \( A \) is shorter than \( B \), 2) \( A \) is similar to \( B \) or 3) \( B \) is shorter than \( A \). If \( \lambda \) signifies the ‘family’ of well-ordered sets similar to a well-ordered set ‘\( A \)’, then \( \lambda \) is called the ‘ordinal’ number of \( A \) i.e.

\[
\lambda \equiv ord(A)
\]

Proposition 14: If \( A \xrightarrow{f} \bar{A} \) is valid and validly ordered, \( A \) is a well-ordered set and \( \lambda = ord(A) \), then \( ord(A) \xrightarrow{r} ord(\bar{A}) \) is valid, \( \bar{\lambda} = ord(\bar{A}) \) and \( r(\lambda) = \bar{\lambda} \).

Proof: For suppose not, then \( A \) would be either longer or shorter than \( \bar{A} \), and hence, \( A \neq \bar{A} \), which violates a necessary condition of a validly ordered system. Therefore,
\[ \text{ord}(A) \overset{f}{\rightarrow} \text{ord}(\bar{A}) \]
is valid.

Proposition 15: Let \( s(\lambda) \) be the set of ordinal numbers that precede the ordinal number '\( \lambda \)' , then

\[ \lambda = \text{ord}(s(\lambda)) \]

Proof: Let \( \lambda = \text{ord}(A) \) and let \( S(A) \) denote the family of all initial segments '\( s(a) \)' of \( A \), ordered by set inclusion. Then \( A \simeq S(A) \) and \( \lambda = \text{ord}(S(A)) \). If \( \mu \in s(\lambda) \), then \( \mu < \lambda \).

But there is a unique initial segment of \( A \) such that \( \mu = \text{ord}(s(a)) \). Hence, the function \( s(\lambda) \rightarrow S(A) \) defined by \( f(\mu) = s(a) \), \( \mu = \text{ord}(s(a)) \) is one-to-one. But \( f \) is onto, since if \( s(b) \in S(A) \), then \( \text{ord}(s(b)) = \eta < \text{ord} (A) = \lambda \). It is easily shown that \( f \) preserves order [179]. This completes the proof.

Roughly speaking, an ordinal number is defined as the set of ordinal numbers which precede it [179]:

\[
\begin{align*}
0 & \equiv \emptyset \\
1 & \equiv \{0\} \\
2 & \equiv \{0,1\} \\
3 & \equiv \{0,1,2\} \\
\vdots \\
\omega & \equiv \{0,1,2, \ldots \} \\
\omega + 1 & \equiv \{0,1,2, \ldots , \omega\} \\
\omega + 2 & \equiv \{0,1,2, \ldots , \omega, \omega + 1\} \\
\vdots \\
\omega 2 & \equiv \{0,1,2, \ldots , \omega, \omega + 1, \ldots \} \\
\omega 2 + 1 & \equiv \{0,1,2, \ldots , \omega, \omega + 1, \omega \ldots , \omega 2\} \\
\vdots
\end{align*}
\]

Note that \( 2 = \text{ord} (\{a, b\}) \) and \( \omega = \text{ord} (\{0,1,2, \ldots \}) \). Hence,

\[
2\omega = \{(a,1), (b,1), (a,2), (b,2), \ldots , (a,n), (b,n), \ldots \} = \omega,
\]
since multiplication of ordinals is defined by

\[
\lambda \mu = \text{ord}(A \times B), \quad \lambda = \text{ord}(A), \quad \mu = \text{ord}(B)
\]

But

\[
\omega 2 = \{(1,a), (2,a), \ldots ; (1,b), (2,b), \ldots \} > \omega
\]

Note that ordinal multiplication is not commutative [179].
2.9 The Laws of Valid Systems

Figure 2.9-1 shows the algebra of valid systems. If \( U \rightarrow \overline{U} \) is a valid system, then all its subsystems are valid. The laws of valid systems are similar to the laws of set operations associated with Cantorian set theory (see Book II: Sec. 3.2.2.1).

<table>
<thead>
<tr>
<th>The Algebra of Valid Systems</th>
</tr>
</thead>
<tbody>
<tr>
<td>( U \rightarrow \overline{U} ), ( A, B, C \subset U )</td>
</tr>
<tr>
<td>Idempotent Laws</td>
</tr>
<tr>
<td>1a. ( A \cup A = A \rightarrow \overline{A} = A \lor \overline{A} )</td>
</tr>
<tr>
<td>1b. ( A \cap A = A \rightarrow \overline{A} = A \land \overline{A} )</td>
</tr>
<tr>
<td>Associative Laws</td>
</tr>
<tr>
<td>2a. ( (A \cup B) \cup C = A \cup (B \cup C) \rightarrow (\overline{A} \lor \overline{B} \lor \overline{C}) = \overline{A} \lor (\overline{B} \lor \overline{C}) )</td>
</tr>
<tr>
<td>2b. ( (A \cap B) \cap C = A \cap (B \cap C) \rightarrow (\overline{A} \land \overline{B} \land \overline{C}) = \overline{A} \land (\overline{B} \land \overline{C}) )</td>
</tr>
<tr>
<td>Commutative Laws</td>
</tr>
<tr>
<td>3a. ( A \cup B = B \cup A \rightarrow \overline{A} \lor \overline{B} = B \lor \overline{A} )</td>
</tr>
<tr>
<td>3b. ( A \cap B = B \cap A \rightarrow \overline{A} \land \overline{B} = B \land \overline{A} )</td>
</tr>
<tr>
<td>Distributive Laws</td>
</tr>
<tr>
<td>4a. ( (A \cup (B \cap C)) = (A \cup B) \cap (A \cup C) \rightarrow (\overline{A} \lor (\overline{B} \land \overline{C})) = (\overline{A} \lor \overline{B}) \land (\overline{A} \lor \overline{C}) )</td>
</tr>
<tr>
<td>4b. ( (A \cap (B \cup C)) = (A \cap B) \cup (A \cap C) \rightarrow (\overline{A} \land (\overline{B} \lor \overline{C})) = (\overline{A} \land \overline{B}) \lor (\overline{A} \land \overline{C}) )</td>
</tr>
<tr>
<td>Identity Laws</td>
</tr>
<tr>
<td>5a. ( A \cup \emptyset = A \rightarrow \overline{A} = A \lor \overline{\emptyset} = A \lor \emptyset )</td>
</tr>
<tr>
<td>5b. ( A \cap U = A \rightarrow \overline{A} = A \land \overline{U} )</td>
</tr>
<tr>
<td>6a. ( A \cup U = U \rightarrow \overline{U} = A \lor \overline{U} )</td>
</tr>
<tr>
<td>6b. ( A \cap \emptyset = \emptyset \rightarrow \overline{\emptyset} = \emptyset \land \overline{\emptyset} )</td>
</tr>
<tr>
<td>Compliment Laws</td>
</tr>
<tr>
<td>7a. ( A \cup \sim A = U \rightarrow \overline{U} = A \lor \overline{A} )</td>
</tr>
<tr>
<td>7b. ( A \cap \sim A = \emptyset \rightarrow \overline{\emptyset} = A \land \overline{\sim A} )</td>
</tr>
<tr>
<td>8a. ( \sim \sim A = A \rightarrow \overline{A} = \sim \overline{A} )</td>
</tr>
<tr>
<td>8b. ( \sim = \emptyset \rightarrow \overline{\emptyset} = \sim \emptyset, \sim \emptyset = U \rightarrow \overline{U} = \overline{\emptyset} )</td>
</tr>
<tr>
<td>De Morgan’s Laws</td>
</tr>
<tr>
<td>9a. ( \sim (A \lor B) = \sim A \land \sim B \rightarrow \overline{A \lor B} = \overline{A} \land \overline{B} )</td>
</tr>
<tr>
<td>9b. ( \sim (A \land B) = \sim A \lor \sim B \rightarrow \overline{A \land B} = \overline{A} \lor \overline{B} )</td>
</tr>
</tbody>
</table>

Table 2.9-1

The importance of valid systems is illustrated by the following example: Suppose \( A \rightarrow \overline{A} \) is valid, but \( B \rightarrow \overline{B} \) is complete and consistent, but invalid. If \( A \cap B = \emptyset \), then

\[
A \cap B = \emptyset \rightarrow \overline{\emptyset} = A \land \overline{\emptyset} = \overline{A} \lor \overline{\emptyset} = A \lor \emptyset = A
\]

which is valid. But \( A \cup B \rightarrow \overline{A \cup B} \) is partially valid, since

\[
\forall a \forall b (a \in A, b \in B \rightarrow r(a) = \overline{\overline{a} \land s(b)} = \overline{\overline{a} \land \overline{b}})
\]

Now suppose \( A \cap B \neq \emptyset \). Suppose further that \( a, b \in A, B \) implies
\[ r(a) = \bar{a}, \quad r(b) = \bar{b}, \quad s(a) = \bar{c}, \quad s(b) = \bar{d}, \quad \bar{a} \neq \bar{c}, \quad \bar{b} \neq \bar{d} \]

Therefore,

\[ \{a, b\} \cap \{a, b\} = \{a, b\} \rightarrow \{\bar{a}, \bar{b}\} \cap \{\bar{c}, \bar{d}\} = \emptyset, \]

which is invalid. On the other hand,

\[ \{a, b\} \cup \{a, b\} = \{a, b\} \rightarrow \{\bar{a}, \bar{b}\} \cup \{\bar{c}, \bar{d}\} = \{\bar{a}, \bar{b}, \bar{c}, \bar{d}\}, \]

which is also invalid.

The upshot is that the laws shown in table 2.9-1 are valid if and only if the operations are performed on valid systems. In other words, set operations performed on valid systems are valid.

### 2.9.1 Valid Operators

Consider a set of operators ‘\( O = \{\cup, \cap, \Delta, \ldots\}\)’ and the set ‘\( \bar{O} = \{\bar{\cup}, \bar{\cap}, \bar{\Delta}, \ldots\}\)’ also of operators. A system ‘\( O \rightarrow \bar{O}\)’ consists of valid operators if and only if

\[ r(\cup) = \bar{\cup}, \quad r(\cap) = \bar{\cap}, \quad r(\Delta) = \bar{\Delta}, \ldots \]

Binary operators are functions i.e.

\[ \forall x \forall y \exists \exists \exists f (x, y, z \in A \rightarrow (x \circ y) \rightarrow z), \]

where ‘\( \circ\)’ is called a ‘binary operator’. A system of binary operators ‘\( O \rightarrow \bar{O}\)’ is valid if and only if

\[ (x \circ y) \rightarrow z \rightarrow (\bar{x} \circ \bar{y}) \rightarrow \bar{z}, \quad f(\circ) = \bar{\circ} \in \bar{O}, \quad \circ \in O \]

In other words, \( \bar{f} \) must be a valid instantiation of \( f \).

### 2.10 Paradoxes

Objective logic does not permit a valid system of all systems (sets that are members of themselves). A formal proof can be given as follows: Let \( S \rightarrow \bar{S} \) be a valid system of systems, where \( S = \{S_1, S_2, \ldots\} \) is the set of all sets. According to objective logic, \( S \) must have an instantiation set ‘\( \bar{S} = \{\bar{S}_1, \bar{S}_2, \ldots\}\)’. But a set of all sets requires that \( \bar{S} \in S \) i.e.

\[ S = \{S_1, S_2, \ldots, \bar{S}_1, \bar{S}_2, \ldots\} \]

36
But each set in ‘\( S \)’ is required to have an instantiation ‘\( \bar{S} \)’ i.e.
\[
\bar{S} \equiv \{\bar{S}_1, \bar{S}_2, ..., \bar{S}_1, \bar{S}_2, ... \}
\]
But then \( S \) and \( \bar{S} \) would have common elements, which violates the definition of a valid system of systems. This should be plain, since a set of all sets necessarily implies a single set system of logic, which is contrary to objective logic.

Example: The barber paradox: All men in a certain town either shave themselves or are shaved by the barber. Let \( A \) be the set of all men in the town and let \( S \) be the set of those men who shave themselves and \( B \) be the set of those men shaved by the barber. The men are split into two mutually exclusive relational sets:
\[
A \times S = \{(x, s)\}, \quad A \times B = \{(x, b)\}
\]
The set ‘\( A \times S \)’ contains those men who shave themselves and the set ‘\( A \times B \)’ those shaved by the barber. The assumption is that
\[
(A \times S) \cap (A \times B) = \emptyset
\]
Let \( M \subset A = \{a, b, c\} \) signify three men who live in the town, where \( b \) designates the ‘barber’. Let \( \langle a, s \rangle \in A \times S \) and \( \langle c, b \rangle \in A \times B \). Hence, \( M \times S = \{(a, s), (b, s)\} \) and \( M \times B = \{(c, b), (b, b)\} \) and let
\[
R' = (M \times S) \cup (M \times B) = \{(a, s), (b, s), (c, b), (b, b)\}
\]
The possible instantiations consist of the following set of relations:
\[
\bar{R} = \{\langle a, \bar{s} \rangle, \langle a, \bar{b} \rangle, \langle a, \bar{c} \rangle, \langle \bar{b}, a \rangle, \langle \bar{b}, \bar{s} \rangle, \langle \bar{b}, \bar{c} \rangle, \langle \bar{c}, a \rangle, \langle \bar{c}, \bar{s} \rangle \}
\]
where \( \langle a, \bar{s} \rangle \) signifies that \( a \) shaves himself and \( \langle a, \bar{b} \rangle \) signifies that \( b \) shaves \( a \) and so on. Hence,
\[
\bar{R}' = \{\langle \bar{a}, \bar{s} \rangle, \langle \bar{b}, \bar{s} \rangle, \langle \bar{c}, \bar{b} \rangle \}
\]
since \( \bar{R}' \) contains only the true relations (assuming the barber shaves). But the system \( R' \rightarrow \bar{R}' \) is either inconsistent or incomplete, since \( \# \bar{R}' \neq \# R' \), and hence, cannot be valid.

By the same token, objective logic does not allow a set of all ordinal or all cardinal numbers, since \( \lambda = \{\lambda_1, \lambda_2, ...\} \), the set of all ordinals, would have an instantiation set ‘\( \bar{\lambda} = \{\bar{\lambda}_1, \bar{\lambda}_2, ...\} \). But then \( \bar{\lambda} \in \lambda \), which is impossible. The same argument holds for the set of all cardinal numbers. By creating a two-set system of logic, the paradoxes associated with Cantorian set theory are avoided.
2.11 Concluding Remarks

This chapter introduced a new system of logic called ‘objective’ logic. It affirms the existence of universal and abstract mathematical objects. The logic is completely defined in terms of two sets, a nominal set ‘\(S\)’, which represents the notion of ‘thoughts’, and a set ‘\(\bar{S}\)’, which represents the instantiation of the thoughts in \(S\). The primary notion within objective logic is the ‘system’, designated ‘\(S \xrightarrow{r} \bar{S}\)’, where \(r\) defines a relationship between the elements belonging \(S\) and \(\bar{S}\) respectively.

If

\[
\forall s \forall \bar{s}(s \in S \land \bar{s} \in \bar{S} \rightarrow r(s) = \bar{s}),
\]

then \(S \xrightarrow{r} \bar{S}\) is called a ‘valid’ system, where \(r(s) = \bar{s}\) is called a ‘statement’. Moreover, if \(s \in S\) and \(r(s) = \bar{s}\), then the statement is ‘true’, otherwise, it is ‘false’. A relation ‘\(r\)’ that is onto, but not a function, represents an inconsistency called a ‘confusion’ – at least one distinct thought is associated with more than one instantiation. A relation ‘\(\bar{r}\)’ that is onto, but not a function, represents an inconsistency called a ‘confounding’ – at least one distinct instantiation is associated with more than one thought. If \(r\) is not onto, the system is ‘incomplete’. An incomplete system contains incomplete statements. If \(\bar{r}\) is not onto, then there exists an element ‘\(s \in S\)’ with no instantiation. This is called a ‘fabrication’.

All valid systems are consistent, complete and true. All subsystems of valid systems are valid. Set operations on systems are valid if and only if the systems are valid.

The set ‘\(R = (A, B, C)\)’ is called a ‘relation’ in \(A\) and \(B\). If \((a, b) \in C\) such that

\[
\langle \bar{a}, \bar{b} \rangle \in \bar{A} \times \bar{B} \rightarrow t(\langle a, b \rangle) = \langle \bar{a}, \bar{b} \rangle,
\]

then the relation ‘\(\langle a, b \rangle\)’ is ‘true’. Otherwise, it is ‘false’.

A relation ‘\(R = (A, A, C)\)’ orders \(A\) if there exist elements ‘\(a, b \in A\)’ such at \(a \lessdot b \in C\). If

\[
\forall a \forall b(a \lessdot b \in C \rightarrow \bar{a} \lessdot \bar{b} \in \bar{C}),
\]

then the system ‘\(R \xrightarrow{t} \bar{R}\)’ is validly ordered. Hence, \(\bar{R}\) represents an instantiation of the ordering in \(A\). Most number systems contain orderings. Number systems will be developed in the next chapter.

Finally, with the development of a two-set system of logic, the paradoxes associated with Cantorian set theory are avoidable. Many more theorems in objective logic could have been proven, but the discussion here serves as an introduction to the subject and will suffice for the deliberations in this book.
Chapter 3

Objective Numbers

“My big thesis is that although the world looks messy and chaotic, if you translate it into the world of numbers and shapes, patterns emerge and you start to understand why things are the way they are.”

- Marcus du Sautoy

3.0 Introduction

Chapter 2 introduced a two-set system of logic, defined in terms of a set ‘S’ related to a set ‘S̅’. The relationship ‘S → S̅’ is called a ‘system’. Fundamental to the relationship between S and S̅, no element of S can be an element of S̅:

∀s∀s̅(s ∈ S ∧ s̅ ∈ S̅ → s ∈ S ∧ s̅ ∈ S̅)

The form ‘r(s_i) = s̅_j, s_i ∈ S, s_i ∈ S’ is called a ‘statement’. A statement can be ‘true’ or ‘false’. A system ‘S → S̅’ is a collection of statements. The system is ‘valid’ if all its statements true and r is one-to-one and onto:

∀s_i ∀s̅_i (s_i ∈ S, s̅_i ∈ S̅ → r(s_i) = s̅_i)

Set operations performed on valid systems result in valid systems. Given a system ‘U → U̅’, if S ⊂ U, then S → S̅ is called a ‘subsystem’ of U → U̅ denoted

S_r → S̅ ⊂ U_r → U̅

Every subsystem of a valid system is valid. A system with at least one true statement is called ‘partially valid’. Every partially valid system has a non-empty valid subsystem. The only valid subsystem of a complete and consistent invalid system is the null system ‘∅ → ∅’. If A is an ordered set, A_r → A̅ has a valid ordering if and only if A ∼ A̅.

This chapter develops instantiated number systems, consistent with the principles of objective logic. The properties of the numbers developed describe valid, consistent and complete number systems.

3.1 The Natural Numbers

The set ‘N = {1,2,3,...}’ is called the ‘natural numbers’. The natural numbers are developed through ‘mathematical induction’. Let S ⊂ N and let

1. 1 ∈ S
2. n ∈ S → n + 1 ∈ S,
then $S = N$.

### 3.1.1 Instantiation of the Natural Numbers

Objective logic requires that $N$ be validated by a set $\bar{N}$. In other words, $N \xrightarrow{r} \bar{N}$ must be a valid system. Hence, there must exist a set $\bar{S} \subseteq \bar{N}$ such that

1. $\bar{1} \in \bar{S}$
2. $\bar{n} \in \bar{S} \rightarrow \bar{n} + \bar{1} \in \bar{S}$,

then $\bar{S} = \bar{N}$.

Suppose $\bar{N} = \{-1, -2, -3, \ldots\}$, where

$$\forall n \in N \rightarrow r(n) = -n \in \bar{N}$$

The set of statements $'r(n) = -n'$ is one-to-one and onto. Hence, $N \xrightarrow{r} \bar{N}$ is consistent and complete. All the statements $'r(n) = \bar{n}'$ are, by fiat, true. Therefore, $N \xrightarrow{r} \bar{N}$ is valid. Moreover, $N$ is a well-ordered set i.e.

$$\forall n \in N \rightarrow n < n + 1$$

and $N$ has a first element $'1'$.

Note that $r(1) = -1$. If 1 is the first element in $N$, then $-1$ must be the first element in $\bar{N}$ and

$$\forall n \in N, n < n + 1 \rightarrow r(n) < r(n + 1) \rightarrow -n < -n + (-1) \in \bar{N}$$

Hence, $\bar{N} \cong N$. Therefore, $(N, N, C) \xrightarrow{f} (\bar{N}, \bar{N}, \bar{C})$ constitutes a valid ordering.

### 3.1.2 Properties of the Natural Numbers

A binary operation $'\circ'$ is valid if and only if

$$\forall x \forall y \exists z (x, y \in S \land x \circ y = z \in S \leftrightarrow \bar{x} \bar{y} \bar{z} = \bar{z} \bar{x} \circ \bar{y} = r(x \circ y) = \bar{x} \bar{y})$$

Hence, $r(\circ) = \bar{\circ}$. 
The natural numbers have the algebraic properties under the operations of ‘addition’ and ‘multiplication’ listed in table 3.1-1a.

<table>
<thead>
<tr>
<th>Abbr.</th>
<th>Name</th>
<th>Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>Closure</td>
<td>$\forall n \forall m(n + m \in N)$</td>
</tr>
<tr>
<td>$A_2$</td>
<td>Commutative Law</td>
<td>$\forall n \forall m(n + m = m + n)$</td>
</tr>
<tr>
<td>$A_3$</td>
<td>Associative Law</td>
<td>$\forall n \forall m \forall p(n + (m + p) = (m + n) + p)$</td>
</tr>
<tr>
<td>$M_1$</td>
<td>Closure</td>
<td>$\forall n \forall m(n \cdot m \in N)$</td>
</tr>
<tr>
<td>$M_2$</td>
<td>Commutative Law</td>
<td>$\forall n \forall m(n \cdot m = m \cdot n)$</td>
</tr>
<tr>
<td>$M_3$</td>
<td>Associative Law</td>
<td>$\forall n \forall m \forall p(n \cdot (m \cdot p) = (m \cdot n) \cdot p)$</td>
</tr>
<tr>
<td>$M_4$</td>
<td>Identity</td>
<td>$\forall n \exists! 1(n \cdot 1 = n)$</td>
</tr>
<tr>
<td>$D_1$</td>
<td>Distributive (Left)</td>
<td>$\forall n \forall m \forall p(m \cdot (n + p) = m \cdot n + m \cdot p)$</td>
</tr>
<tr>
<td>$D_2$</td>
<td>Distributive (Right)</td>
<td>$\forall n \forall m \forall p((n + p) \cdot m = n \cdot m + p \cdot m)$</td>
</tr>
</tbody>
</table>

Table 3.1-1a

The properties of $\overline{N}$ are listed in table 3.1-1b.

<table>
<thead>
<tr>
<th>Abbr.</th>
<th>Name</th>
<th>Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>Closure</td>
<td>$\forall (\neg n) \forall (\neg m)((\neg n + (\neg m) \in \overline{N})$</td>
</tr>
<tr>
<td>$\bar{A}_2$</td>
<td>Commutative Law</td>
<td>$\forall (\neg n) \forall (\neg m)((\neg n + (\neg m) = (\neg m + (\neg n))$</td>
</tr>
<tr>
<td>$\bar{A}_3$</td>
<td>Associative Law</td>
<td>$\forall (\neg n) \forall (\neg m)\forall (\neg p)((\neg n + (\neg m + (\neg p))$ $= (\neg n + (\neg m)) + (\neg p)$</td>
</tr>
<tr>
<td>$\bar{M}_1$</td>
<td>Closure</td>
<td>$\forall (\neg n) \forall (\neg m)((\neg n \cdot (\neg m) \in \overline{N})$</td>
</tr>
<tr>
<td>$\bar{M}_2$</td>
<td>Commutative Law</td>
<td>$\forall (\neg n) \forall (\neg m)((\neg n \cdot (\neg m) = (\neg m \cdot (\neg n))$</td>
</tr>
<tr>
<td>$\bar{M}_3$</td>
<td>Associative Law</td>
<td>$\forall (\neg n) \forall (\neg m)\forall (\neg p)((\neg n \cdot (\neg m \cdot (\neg p))$ $= (\neg n \cdot (\neg m)) \cdot (\neg p)$</td>
</tr>
<tr>
<td>$\bar{M}_4$</td>
<td>Identity</td>
<td>$\forall (\neg n) \exists! (\neg 1)((\neg n \cdot (\neg 1) = (\neg n))$</td>
</tr>
<tr>
<td>$\bar{D}_1$</td>
<td>Distributive (Left)</td>
<td>$\forall (\neg n) \forall (\neg m)\forall (\neg p)[((\neg m \cdot (\neg n + (\neg p)))]$ $= (\neg m \cdot (\neg n) + (\neg m \cdot (\neg p))]$</td>
</tr>
<tr>
<td>$\bar{D}_2$</td>
<td>Distributive (Right)</td>
<td>$\forall (\neg n) \forall (\neg m)\forall (\neg p)[((\neg n + (\neg p)) \cdot (\neg m)$ $= (\neg n \cdot (\neg m) + (\neg p \cdot (\neg m))]$</td>
</tr>
</tbody>
</table>

Table 3.1-1b
To ensure closure in $\bar{N}$, unlike ordinary arithmetic, multiplying two negative numbers results in a negative number. The properties of $N$ and $\bar{N}$ are in one-to-one correspondence. For example,

$$\forall n \forall m (n, m \in N \land (n + m) \in N \rightarrow (-n + (-m)) \in \bar{N})$$

Moreover,

$$n + m = p \in N \rightarrow -n + (-m) = -p \in \bar{N} \rightarrow r(p) = -p = \bar{p}$$

The same can be shown for all the properties listed in tables 3.1-1a and b. Hence,

$$N \not\rightarrow -N$$

is a valid and validly ordered system.

### 3.2 The Integers

The set \(Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \}\) is called the 'integers'. It is desirable to construct sets of numbers from previously defined sets of numbers.

#### 3.2.1 Instantiating the Integers

Define

$$\sqrt{n} \cdot \sqrt{n} = \sqrt{n \cdot n} = n, \quad n \in N$$

Developing the integers requires an additive identity element, denoted \(\sqrt{0}\) \((\sqrt{n} = \sqrt{n + \sqrt{0}})\) and an additive inverse \(-\sqrt{n}\) \((\sqrt{n} + (-\sqrt{n}) = \sqrt{0})\):

$$\sqrt{n} + (-\sqrt{n}) = \sqrt{0} \rightarrow -\sqrt{n} + \sqrt{n} + (-\sqrt{n}) = \sqrt{0} - \sqrt{n} = -\sqrt{n} \rightarrow -\sqrt{n} + \left(-(-\sqrt{n})\right) = \sqrt{0}$$

$$\rightarrow \sqrt{n} + (-\sqrt{n}) + \left(-(-\sqrt{n})\right) = \sqrt{n} \rightarrow \left(-(-\sqrt{n})\right) = \sqrt{n} \rightarrow \left(-(-\sqrt{n} \cdot \sqrt{n})\right)$$

$$= \sqrt{n} \cdot \sqrt{n} \rightarrow (-(-n)) = n$$

Moreover,

$$-\sqrt{n} \cdot \sqrt{n} = -n, \quad n \in N:$$

$$\sqrt{0} + \sqrt{0} = \sqrt{0} \rightarrow \sqrt{n} \cdot \sqrt{0} + \sqrt{n} \cdot \sqrt{0} = \sqrt{n} \cdot \sqrt{0} \rightarrow \sqrt{n} \cdot \sqrt{0} = \sqrt{0} \rightarrow \sqrt{n} \cdot \sqrt{n} + \sqrt{n} \cdot (-\sqrt{n})$$

$$= \sqrt{n} \cdot \sqrt{0} = \sqrt{0} \rightarrow -\sqrt{n} \cdot (-\sqrt{n}) + \sqrt{n} \cdot \sqrt{n} + \sqrt{n} \cdot (-\sqrt{n})$$

$$= \sqrt{0} + \left(-\sqrt{n} \cdot (-\sqrt{n})\right) \rightarrow \sqrt{n} \cdot \sqrt{n} = -\sqrt{n} \cdot (-\sqrt{n}) \rightarrow -\sqrt{n} \cdot \sqrt{n}$$

$$= -(-\sqrt{n}) \cdot (-\sqrt{n}) \rightarrow -n = \sqrt{n} \cdot (-\sqrt{n}),$$

Note that multiplying a negative number by another negative number results in a positive number and multiplying a negative number by a positive number results in a negative number. This restores accepted features of the integers.
Hence, let
\[ \sqrt{Z} = \{ \pm \sqrt{n} \} \rightarrow Z = \{ \pm \sqrt{n} \cdot \sqrt{n} = \pm n \}, \quad n \in N, \quad 0 \in Z, \]
and let
\[ \sqrt{Z} = \{ \pm \sqrt{-n} \} \rightarrow \bar{Z} = \{ \pm \sqrt{-n} \cdot \sqrt{-n} = \sqrt{-n} \cdot (-n) = \pm \sqrt{-n \cdot n} = \pm in \}, \]
\[ -n \in \bar{N}, \quad i = \sqrt{-1}, \quad i0 = 0 \in \bar{Z} \]

In this case, there are two possible valid systems ‘\( Z \rightarrow \bar{Z} \)’. In the first case, \( f(n) = in \) for all \( n \), where \( \pm n \in Z \sim 0 \) and \( \pm in \in Z \sim 0 \). Each of the pairs ‘\( (n, n) \)’ lie along the red line in fig. 3.2.1-1. In the second case, \( f(-n) = in \) for all \( n \). Each of the pairs ‘\( (-n, n) \)’ lie along the blue line in fig. 3.2.1-1.

![Figure 3.2.1-1](image)

In both cases, \( f \) is one-to-one and onto. Both systems are complete and consistent. Moreover, in each system, considered separately,
\[ \forall n(n \in Z \sim 0 \rightarrow f(n) = \bar{n}) \]

Hence, \( Z \sim 0 \rightarrow \bar{Z} \sim 0 \) is valid in each case.

The set ‘\( Z \sim 0 \)’ is totally ordered by virtue of the order in \( N \). Hence,
\[ \forall n(n < n + 1 \in Z \sim 0 \rightarrow f(n) < f(n + 1) = \bar{n} < \bar{n} + 1 \in \bar{Z} \sim 0) \]
If \( f(n) = \bar{n} \), then
\[ f(n + 1) = f(n) + f(1) = \bar{n} + 1 \]
for all \( n \) and \( \bar{n} \). Note that \( f \) is linear. Therefore, \( Z \sim 0 \simeq \bar{Z} \sim 0 \) in both systems, and hence, the systems are validly ordered.
Finally, define a number '0 ∈ Z' such that \( f(0) = i0 = 0 \in \mathbb{Z} \). Note that the pair ‘(0, 0)’ is a member of both the red and blue systems (see fig. 3.2.1-1) and is the only pair belonging to both systems. Therefore, \( 
abla \to i\mathbb{Z} \) is a valid and validly ordered system.

### 3.2.2 Properties of the Integers

The integers have the algebraic properties listed in table 3.2.2-1a.

<table>
<thead>
<tr>
<th>Abbr.</th>
<th>Name</th>
<th>Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_1 )</td>
<td>Closure</td>
<td>( \forall n \forall m(n + m \in \mathbb{Z}) )</td>
</tr>
<tr>
<td>( A_2 )</td>
<td>Commutative Law</td>
<td>( \forall n \forall m(n + m = m + n) )</td>
</tr>
<tr>
<td>( A_3 )</td>
<td>Associative Law</td>
<td>( \forall n \forall m \forall p(n + (m + p) = (m + n) + p) )</td>
</tr>
<tr>
<td>( A_4 )</td>
<td>Cancellation Law</td>
<td>( \forall n \forall m \forall p(n + p = m + p \to n = m) )</td>
</tr>
<tr>
<td>( A_5 )</td>
<td>Identity</td>
<td>( \forall n \exists! 0(n + 0 = n) )</td>
</tr>
<tr>
<td>( A_6 )</td>
<td>Inverse</td>
<td>( \forall n(n + (−n) = 0) )</td>
</tr>
<tr>
<td>( M_1 )</td>
<td>Closure</td>
<td>( \forall n \forall m(n \cdot m \in \mathbb{Z}) )</td>
</tr>
<tr>
<td>( M_2 )</td>
<td>Commutative Law</td>
<td>( \forall n \forall m(n \cdot m = m \cdot n) )</td>
</tr>
<tr>
<td>( M_3 )</td>
<td>Associative Law</td>
<td>( \forall n \forall m \forall p(n \cdot (m \cdot p) = (m \cdot n) \cdot p) )</td>
</tr>
<tr>
<td>( M_4 )</td>
<td>Identity</td>
<td>( \forall n \exists! 1(n \cdot 1 = n) )</td>
</tr>
<tr>
<td>( D_1 )</td>
<td>Distributive (Left)</td>
<td>( \forall n \forall m \forall p(m \cdot (n + p) = m \cdot n + m \cdot p) )</td>
</tr>
<tr>
<td>( D_2 )</td>
<td>Distributive (Right)</td>
<td>( \forall n \forall m \forall p((n + p) \cdot m = n \cdot m + p \cdot m) )</td>
</tr>
</tbody>
</table>

Table 3.2.2-1a

The properties of \( \mathbb{Z} \) are shown in table 3.2.2-1b.

<table>
<thead>
<tr>
<th>Abbr.</th>
<th>Name</th>
<th>Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{A}_1 )</td>
<td>Closure</td>
<td>( \forall in \forall im(in + im \in \mathbb{Z}) )</td>
</tr>
<tr>
<td>( \bar{A}_2 )</td>
<td>Commutative Law</td>
<td>( \forall in \forall im(in + im = im + in) )</td>
</tr>
<tr>
<td>( \bar{A}_3 )</td>
<td>Associative Law</td>
<td>( \forall in \forall im \forall ip(in + (im + ip) = (im + in) + ip) )</td>
</tr>
<tr>
<td>( \bar{A}_4 )</td>
<td>Cancellation Law</td>
<td>( \forall in \forall im \forall ip(in + ip = im + ip \to in = im) )</td>
</tr>
<tr>
<td>( \bar{A}_5 )</td>
<td>Identity</td>
<td>( \forall in \exists! 0(in + i0 = in) )</td>
</tr>
<tr>
<td>( \bar{M}_1 )</td>
<td>Closure</td>
<td>( \forall in \forall im(in \cdot im \in \mathbb{Z}) )</td>
</tr>
<tr>
<td>( \bar{M}_2 )</td>
<td>Commutative Law</td>
<td>( \forall in \forall im(in \cdot im = im \cdot in) )</td>
</tr>
<tr>
<td>( \bar{M}_3 )</td>
<td>Associative Law</td>
<td>( \forall in \forall im \forall ip(in \cdot (im \cdot ip) = (im \cdot in) \cdot ip) )</td>
</tr>
<tr>
<td>( \bar{M}_4 )</td>
<td>Identity</td>
<td>( \forall in \exists! (in \in \mathbb{Z} \to in \cdot i = in) )</td>
</tr>
<tr>
<td>( \bar{D}_1 )</td>
<td>Distributive (Left)</td>
<td>( \forall in \forall im \forall ip(im \cdot (in + ip) = im \cdot in + im \cdot ip) )</td>
</tr>
<tr>
<td>( \bar{D}_2 )</td>
<td>Distributive (Right)</td>
<td>( \forall in \forall im \forall ip((in + ip) \cdot im = in \cdot im + ip \cdot im) )</td>
</tr>
</tbody>
</table>

Table 3.2.2-1b
The rule for multiplying two negative numbers to produce a positive number has been restored. However, multiplying two imaginary numbers results in an imaginary number, which guarantees closure within $\mathbb{Z}$. Multiplication of imaginary numbers can be thought of in the following way: define a number $'i'$ such that

$$\frac{i + i + \cdots + i}{n} = in$$

and

$$\frac{in + in + \cdots + in}{m} = i(n \cdot m) = in \cdot im$$

Therefore, $i$ plays the role of a multiplicative inverse i.e. $i \cdot i = i$. To see this,

$$-1 \in \overline{N} \rightarrow (-1) \cdot (-1) = -1 \cdot 1 = -1 \in \overline{N}$$

Hence,

$$-1 = -1 \rightarrow \sqrt{-1} = i \rightarrow (-1) \cdot (-1) = -1 \cdot 1 = -1 \rightarrow \sqrt{-1} \cdot \sqrt{-1} = i \cdot i = \sqrt{-1} = i$$

Note that the above derivation would not be true if $i$ was treated as an ordinary imaginary number, where $i \cdot i = -1$.

The properties of $\mathbb{Z}$ and $\overline{Z}$ are in one-to-one and onto correspondence. For example,

$$\forall n \forall m (n, m \in \mathbb{Z} \land n + m \in \mathbb{Z} \rightarrow in + im \in \overline{Z})$$

Moreover,

$$n + m = p \in \mathbb{Z} \rightarrow in + im = i(n + m) = ip \in \overline{Z} \rightarrow r(p) = ip = \bar{p}$$

The same can be shown for all the properties listed in table 3.2.2-1a and b.

### 3.3 The Rational Numbers

The set $'R_+^Q = \{n/m\}, n, m \in N'$ is called the set of ‘positive’ rational numbers.

#### 3.3.1 Instantiating the Rational Numbers

Define

$$\sqrt{R_Q} = \left\{ \pm \frac{\sqrt{n}}{\sqrt{m}}, 0 \right\}, \quad n, m \in N, \quad 0 \in Z,$$
and let

\[ \sqrt{R_Q} \equiv \left\{ \pm \sqrt{-\frac{n}{m}} = \pm \sqrt{-n \cdot \left( -\frac{1}{m} \right)} = \pm \sqrt{-\frac{n}{m}} = \pm i \sqrt{\frac{n}{m}}, \quad i \sqrt{0} \right\}, \quad -1, -m, -n \in \bar{N}, \]

\[ i = \sqrt{-1}, \quad i0 \in \bar{Z} \]

In general, \( \pm \sqrt{n/m}, \ n, m \in N \) is not a rational number. Hence, define

\[ R_Q = \left\{ \pm \sqrt{\frac{n}{m}}, \sqrt{0} \cdot \sqrt{0} = \pm \frac{n}{m}, 0 \right\}, \quad n, m \in N, \]

\[ R_Q = \left\{ \pm \sqrt{-n \cdot \left( -\frac{1}{m} \right)} \cdot \sqrt{-n \cdot \left( -\frac{1}{m} \right)}, i \sqrt{0} \cdot i \sqrt{0} = \pm \sqrt{-\frac{n}{m}} \cdot \sqrt{-\frac{n}{m}}, i \sqrt{0} \cdot i \sqrt{0} \right\}, \quad -1, -m, -n \in \bar{N}, \quad i = \sqrt{-1}, \]

\[ i \cdot i = i \]

Again, there are two possible valid systems ‘\( R_Q \xrightarrow{f} \bar{R}_Q \)’. In the first case,

\[ \forall m \forall n \left( m, n \in N \rightarrow f \left( \frac{n}{m} \right) = i \frac{n}{m} \in \bar{R}_Q \right), \]

where each pair lies along the red line in fig. 3.2.1-1. In the second case,

\[ f \left( \frac{-n}{m} \right) = i \frac{n}{m}, \]

each pair lying along the blue line. Unlike the integers, the systems ‘\( R_Q \xrightarrow{f} \bar{R}_Q \)’ are ‘numerically incomplete’, since rational numbers have limits, but the limit points are not rational numbers.
3.3.2 Properties of the Rational Numbers

The rational numbers have the algebraic properties listed in table 3.3.2-1a.

<table>
<thead>
<tr>
<th>Abbr.</th>
<th>Name</th>
<th>Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>Closure</td>
<td>$\forall n \forall m (n + m \in R_Q)$</td>
</tr>
<tr>
<td>$A_2$</td>
<td>Commutative Law</td>
<td>$\forall n \forall m (n \cdot m = m \cdot n)$</td>
</tr>
<tr>
<td>$A_3$</td>
<td>Associative Law</td>
<td>$\forall n \forall m \forall p (n + (m + p) = (m + n) + p)$</td>
</tr>
<tr>
<td>$A_4$</td>
<td>Cancellation Law</td>
<td>$\forall n \forall m \forall p (n + p = m + p \rightarrow n = m)$</td>
</tr>
<tr>
<td>$A_5$</td>
<td>Identity</td>
<td>$\forall n \exists 0 (n + 0 = n)$</td>
</tr>
<tr>
<td>$A_6$</td>
<td>Inverse</td>
<td>$\forall n (n + (-n) = 0)$</td>
</tr>
<tr>
<td>$M_1$</td>
<td>Closure</td>
<td>$\forall n \forall m (n \cdot m \in R_Q)$</td>
</tr>
<tr>
<td>$M_2$</td>
<td>Commutative Law</td>
<td>$\forall n \forall m (n \cdot m = m \cdot n)$</td>
</tr>
<tr>
<td>$M_3$</td>
<td>Associative Law</td>
<td>$\forall n \forall m \forall p (n \cdot (m \cdot p) = (m \cdot n) \cdot p)$</td>
</tr>
<tr>
<td>$M_4$</td>
<td>Cancellation Law</td>
<td>$\forall n \forall m \forall p (n \cdot p = m \cdot p \land p \neq 0 \rightarrow n = m)$</td>
</tr>
<tr>
<td>$M_5$</td>
<td>Identity</td>
<td>$\forall n \exists 1 (n \cdot 1 = n)$</td>
</tr>
<tr>
<td>$M_6$</td>
<td>Inverse</td>
<td>$\forall n \exists n^{-1} (n \neq 0 \rightarrow n \cdot n^{-1} = 1)$</td>
</tr>
<tr>
<td>$D_1$</td>
<td>Distributive (Left)</td>
<td>$\forall n \forall m \forall p (m \cdot (n + p) = m \cdot n + m \cdot p)$</td>
</tr>
<tr>
<td>$D_2$</td>
<td>Distributive (Right)</td>
<td>$\forall n \forall m \forall p ((n + p) \cdot m = n \cdot m + p \cdot m)$</td>
</tr>
</tbody>
</table>

Table 3.3.2-1a

The properties of $\bar{R}_Q$ are listed in table 3.3.2-1b.

<table>
<thead>
<tr>
<th>Abbr.</th>
<th>Name</th>
<th>Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{A}_1$</td>
<td>Closure</td>
<td>$\forall i \forall n \forall m (i \cdot (n + m) \in \bar{R}_Q)$</td>
</tr>
<tr>
<td>$\bar{A}_2$</td>
<td>Commutative Law</td>
<td>$\forall i \forall n \forall m (i \cdot (n + m) = i \cdot (m + n))$</td>
</tr>
<tr>
<td>$\bar{A}_3$</td>
<td>Associative Law</td>
<td>$\forall i \forall n \forall m \forall p (i \cdot (n + m + p) = (i \cdot (n + m)) + p)$</td>
</tr>
<tr>
<td>$\bar{A}_4$</td>
<td>Cancellation Law</td>
<td>$\forall i \forall n \forall m \forall p (i \cdot (n + p) = i \cdot (m + p) \rightarrow i \cdot n = i \cdot m)$</td>
</tr>
<tr>
<td>$\bar{A}_5$</td>
<td>Identity</td>
<td>$\forall i \exists 0 (i \cdot 0 = i)$</td>
</tr>
<tr>
<td>$\bar{A}_6$</td>
<td>Inverse</td>
<td>$\forall i (i + (-i) = 0)$</td>
</tr>
<tr>
<td>$\bar{M}_1$</td>
<td>Closure</td>
<td>$\forall i \forall n \forall m (i \cdot (n \cdot m) \in \bar{R}_Q)$</td>
</tr>
<tr>
<td>$\bar{M}_2$</td>
<td>Commutative Law</td>
<td>$\forall i \forall n \forall m (i \cdot (n \cdot m) = i \cdot (m \cdot n))$</td>
</tr>
<tr>
<td>$\bar{M}_3$</td>
<td>Associative Law</td>
<td>$\forall i \forall n \forall m \forall p (i \cdot (n \cdot (m \cdot p)) = (i \cdot (n \cdot m)) \cdot p)$</td>
</tr>
<tr>
<td>$\bar{M}_4$</td>
<td>Cancellation Law</td>
<td>$\forall i \forall n \forall m \forall p (i \cdot (n \cdot p) = i \cdot (m \cdot p) \land p \neq i \cdot 0 \rightarrow i \cdot n = i \cdot m)$</td>
</tr>
<tr>
<td>$\bar{M}_5$</td>
<td>Identity</td>
<td>$\forall i \exists 1 (i \cdot 1 = i)$</td>
</tr>
<tr>
<td>$\bar{M}_6$</td>
<td>Inverse</td>
<td>$\forall i (i \neq i \cdot 0 \rightarrow i \cdot \frac{i}{n} = i)$</td>
</tr>
<tr>
<td>$\bar{D}_1$</td>
<td>Distributive (Left)</td>
<td>$\forall i \forall n \forall m \forall p (i \cdot (n + p) = i \cdot (m + i \cdot m))$</td>
</tr>
<tr>
<td>$\bar{D}_2$</td>
<td>Distributive (Right)</td>
<td>$\forall i \forall n \forall m \forall p ((i + p) \cdot i \cdot m = i \cdot (i \cdot m + i \cdot p \cdot i))$</td>
</tr>
</tbody>
</table>

Table 3.3.2-1b
Like \( \mathbb{Z} \), multiplying two imaginary rational numbers results in an imaginary rational number. This guarantees closure within \( \mathbb{R}_Q \). The number ‘\( i \)’ plays the same role in \( \mathbb{R}_Q \) as it does in \( \mathbb{Z} \), the ‘multiplicative identity’.

If \( s/m \in \mathbb{R}_Q \), then \( f(s/m) = is/m \in \mathbb{R}_Q \). And if \( t/n \in \mathbb{R}_Q \), then
\[
f\left(\frac{s}{m} + \frac{t}{n}\right) = f\left(\frac{s}{m}\right) + f\left(\frac{t}{n}\right) = i\left(\frac{s}{m}\right) + i\left(\frac{t}{n}\right) = i \left(\frac{s \cdot in + m \cdot t}{m \cdot n}\right)
\]
and
\[
f\left(\frac{s \cdot t}{m \cdot n}\right) = f\left(\frac{s}{m}\right) \cdot f\left(\frac{t}{n}\right) = i\left(\frac{s}{m}\right) \cdot i\left(\frac{t}{n}\right) = i \frac{s \cdot t}{m \cdot n}
\]

The set of rational numbers is totally ordered. The ordering relations are defined as follows:
\[
\forall x \forall y (x, y \in \mathbb{R}_Q \land x < y \leftrightarrow x - y < 0) \\
\forall x \forall y (x, y \in \mathbb{R}_Q \land x > y \leftrightarrow x - y > 0) \\
\forall x \forall y (x, y \in \mathbb{R}_Q \land x = y \leftrightarrow x - y = 0)
\]
The instantiation set ‘\( \mathbb{R}_Q \)’ is ordered in the same way as \( \mathbb{R}_Q \). Hence,
\[
\mathbb{R}_Q \rightarrow i\mathbb{R}_Q
\]
is a valid and validly ordered system.

### 3.4 The Real Numbers

The real numbers consist of the union of the rational with the irrational numbers. The algebraic properties of the real numbers are identical to those of the rational numbers. The only difference between rational and real numbers is that real numbers contain all their limit numbers, which makes them numerically complete. The systems ‘\( \mathbb{R} \rightarrow \mathbb{R} \)’ of real numbers are developed in identical fashion to the rational numbers. Suffice it to say that \( \mathbb{R} \) instantiates \( \mathbb{R} \) in exactly the same way as \( \mathbb{R}_Q \) instantiates \( \mathbb{R}_Q \):
\[
\mathbb{R} \rightarrow i\mathbb{R}
\]
The two valid systems ‘\( \mathbb{R} \rightarrow \mathbb{R} \)’ are represented by the red and blue lines in fig. 3.2.1-1 respectfully.
However, it could legitimately be argued that the way in which the number systems have been developed results in nothing more than number systems that are ‘isomorphic’ to one another, and hence, simply represent the same system. In other words, \( N \) is isomorphic to \( \bar{N} \) and \( Z \) is isomorphic to \( \bar{Z} \) and so on. Why, then, is it necessary to develop two distinct number systems for each class of numbers? While somewhat valid, that argument does not extend to the ‘complex’ numbers, which will be developed subsequently.

### 3.5 Vector Spaces as Valid Spaces

For convenience, the rules for vector spaces are rewritten here:

1. \((|A| + |B|) + |C| = |A| + (|B| + |C|)\)
2. \(|A| + |0| = |A|\)
3. \(|A| - |A| = |0|\)
4. \(|A| + |B| = |B| + |A|\)
5. \(k \cdot (|B| + |A|) = k \cdot |A| + k \cdot |B|\)
6. \((h + k) \cdot |A| = h \cdot |A| + k \cdot |A|\)
7. \((h \cdot k) \cdot |A| = h \cdot (k \cdot |A|)\)
8. \(1 \cdot |A| = |A| \text{ and } 0 \cdot |A| = |0|\),

where \(k, h, 1\) and \(0\) are scalars which belong to a field ‘\(K\)’. Note that for any points ‘\((n, n)\)’ and ‘\((m, m)\)’ lying along the red line (fig. 3.2.1-1)

\[
\langle n, n \rangle + \langle m, m \rangle = \langle n + m, n + m \rangle,
\]

which also lies along the red line i.e.

\[
f(n + m) = f(n) + f(m) = in + im = i(n + m)
\]

The same can be said of addition along the blue line i.e.

\[
f(-n + m) = f(-n) + f(m) = in + (-im)
\]

The upshot is that vector addition is a valid operation within objective mathematics. Note that

\[
k(m + n) = km + kn \Rightarrow kf(m + n) = kf(m) + kf(n) = ikm + ikn = ik(n + m), \quad k \in K
\]

Hence, scalar multiplication is also a valid operation within objective mathematics, which implies that the rules for vector spaces are valid within objective mathematics.

#### 3.5.1 Hilbert Spaces

A Hilbert space is a vector space ‘\(H\)’ with an inner product ‘\(\langle f|g \rangle\)’ such that the ‘norm’ defined by

\[
| |f| | = \sqrt{\langle f|f \rangle}, \quad |f\rangle \in H
\]
turns $H$ into a complete metric space – a space where an abstract measure for ‘distance’ is defined. If the metric, defined by the norm, is not numerically complete, then $H$ is instead called an ‘inner product’ space [274]. To the extent that a Hilbert space is a vector space, the operations of ‘vector addition’ and ‘scalar multiplication’ are valid operations within objective mathematics. But the inner product of two vectors is an invalid operation.

To see this, suppose $a, b \in C$, where $a = \langle x, y \rangle, \ b = \langle w, z \rangle$. Ordinarily, the inner product $\langle a \vert b \rangle$ is defined

$$\langle a \vert b \rangle = \langle xw + yz, yw - xz \rangle$$

In order for this operation to be valid within objective mathematics

$$f(xw + yz) = i(xw + yz),$$

which is not the case. For example, let $\vert a \rangle = \langle 0, r' \rangle$ and $\vert b \rangle = \langle 0, r'' \rangle$, where both vectors lie along the imaginary axis, $r', r'' \neq 0$. By the ordinary definition of an inner product,

$$x, w = 0, \quad y = r', \quad z = r'' \rightarrow \langle x, y \rangle \langle w, z \rangle = \langle xw + yz, yw - xz \rangle \rightarrow \langle 0, r' \rangle \langle 0, r'' \rangle$$

$$= \langle 0 + r'r'', 0 - 0 \rangle = \langle r'r'', 0 \rangle,$$

where $r'r'' \in R$. In this case, the inner product of the two purely imaginary numbers $\langle a \rangle = \langle 0, r' \rangle$ and $\langle b \rangle = \langle 0, r'' \rangle$ is transformed into a real number.

From the perspective of objective mathematics, $i0$ would instantiate 0 and also instantiate $r'r'' \neq 0$. Therefore, two nominal numbers would have the same instantiation, an inconsistency. Such a system is invalid. There is no provision in objective mathematics where an element in the nominal set can be transformed into an element in the instantiation set and vice versa.

The inner product in objective mathematics is defined

$$\langle n \vert \bar{n} \rangle \cdot \langle m \vert \bar{m} \rangle = \langle n \cdot m \vert \bar{n} \cdot \bar{m} \rangle, \quad n \cdot m \in R, \quad \bar{n} \cdot \bar{m} \in \mathbb{R}$$

If $\langle n \vert \bar{n} \rangle$ and $\langle m \vert \bar{m} \rangle$ lie along the red line in fig. 3.2.1-1, then so does $\langle n \cdot m \vert \bar{n} \cdot \bar{m} \rangle$. In other words,

$$\langle (0, r') \vert (0, r'') \rangle \in \mathbb{Z}$$

The upshot is that objective numbers have, as yet, no defined metric.

3.6 Establishing the Complex Plane

Referring to fig. 3.2.1-1, any number $\langle n, n \rangle$ lying along the red line can be represented in polar coordinates i.e.

$$\langle n, n \rangle = r(\cos(\pi/4), \sin(\pi/4)), \quad r = \sqrt{n^2 + n^2} = \sqrt{2}n \in R,$$
where $r$ can be thought of as a scalar which gives a magnitude to a unit vector $(\cos(\pi/4), \sin(\pi/4))$. Every number along the blue line can be represented by

$$\langle -m, m \rangle = r' \langle \cos(3\pi/4), \sin(3\pi/4) \rangle, \quad r' = \sqrt{2}m \in \mathbb{R}$$

Since the red and blue lines are perpendicular, a rotation by $-\pi/4$ aligns them with the horizontal and vertical axis respectively. Hence,

$$\langle r, 0 \rangle + \langle 0, r' \rangle = \langle r \cos(0), r \sin(0) \rangle + \langle r' \cos(\pi/2), r' \sin(\pi/2) \rangle = \langle r, r' \rangle, \quad \cos(0) = 1, \quad \cos(\pi/2) = 0, \quad \sin(\pi/2) = 1, \quad \sin(0) = 0$$

All points in the complex plane can be represent by the pairs '$(r, r')$', where $r \in \mathbb{R}$ and $ir' \in \overline{\mathbb{R}}$.

### 3.6.1 Complex Numbers

Complex numbers, treated objectively, require that complex number arithmetic be restored, where $i^2 = -1$. In addition, complex numbers must be instantiated.

#### 3.6.1.1 Establishing a Metric in Objective Space

Let $\sqrt{\mathbb{R}} = \{\pm \sqrt{x}\}$, $x \geq 0 \in \mathbb{R}$ and let

$$\sqrt{\mathbb{R}} = \{\pm \sqrt{-x} = \pm i\sqrt{x}\}, \quad x \geq 0 \in \mathbb{R}, \quad ii = \sqrt{-1} = i$$

Note that $\sqrt{\mathbb{R}} \overset{h}{\rightarrow} \sqrt{\mathbb{R}}$ is a valid system if $h(\sqrt{x}) = i\sqrt{x} \in \sqrt{\mathbb{R}}$.

Define a vector '$|\sqrt{r} \rangle = |\sqrt{x} \rangle + |\sqrt{y} \rangle$', which can be given a magnitude (see fig. 3.6.1.1-1):

$$r = (\sqrt{x})^2 + (\sqrt{y})^2 = x + y \rightarrow \sqrt{r} = \pm \sqrt{x + y} \in \sqrt{\mathbb{R}}, \quad x, y \geq 0 \in \mathbb{R}, \quad \sqrt{x}, \sqrt{y} \in \sqrt{\mathbb{R}}$$

and

$$i\sqrt{r} = (i\sqrt{x})^2 + (i\sqrt{y})^2 \rightarrow ir = ix + iy \rightarrow i\sqrt{r} = \pm i\sqrt{x + y} \in \sqrt{\mathbb{R}}, \quad i\sqrt{x}, i\sqrt{y} \in \sqrt{\mathbb{R}}, \quad i \cdot i = i$$

![Figure 3.6.1.1-1](image-url)
The number \( \sqrt{r} \) is called the 'metric' in objective space and is instantiated by \( i \sqrt{r} \). Now let

\[
\langle \sqrt{n}, \sqrt{n} \rangle = \sqrt{r} \langle \cos(\pi/4), \sin(\pi/4) \rangle,
\]

\[
\langle -\sqrt{n}, \sqrt{n} \rangle = \sqrt{r} \langle \cos(3\pi/4), \sin(3\pi/4) \rangle, \quad n \geq 0 \in R, \quad \sqrt{r} = \sqrt{x + y} \in \sqrt{R}, \quad x = n, \quad y = n,
\]

where \( \langle \sqrt{n}, \sqrt{n} \rangle \) and \( \langle -\sqrt{n}, \sqrt{n} \rangle \) lie along the red and blue lines in fig. 3.2.1 respectfully. A rotation of \(-\pi/4\) of the red and blue lines leaves

\[
\langle \sqrt{n}, \sqrt{n} \rangle \rightarrow \langle \sqrt{r}, 0 \rangle, \quad \langle -\sqrt{n}, \sqrt{n} \rangle \rightarrow \langle 0, \sqrt{r} \rangle \rightarrow \langle \sqrt{r}, 0 \rangle + \langle 0, \sqrt{r} \rangle = \langle \sqrt{r}, \sqrt{r} \rangle
\]

Note that \( \langle \sqrt{r}, \sqrt{r} \rangle \) lies along the red line in fig. 3.2.1. Moreover,

\[
\langle \sqrt{r} \sqrt{r}, \sqrt{r} \sqrt{r} \rangle = \langle r, r \rangle = \langle \sqrt{x + y}, \sqrt{x + y}, \sqrt{x + y} \sqrt{x + y} \rangle
= \langle x^2 + y^2 + 2xy, \sqrt{x^2 + y^2 + 2xy}, \sqrt{x^2 + y^2 + 2xy} \rangle, \quad x, y \geq 0
\]

### 3.6.2 Objective Complex Numbers

Defining the complex numbers within objective mathematics encounters two obstacles: 1) the numerical operations defined within ordinary complex number arithmetic are generally incompatible with objective logic; 2) it is not obvious what numbers should instantiate the complex numbers. Recall that complex number arithmetic involves two operations, ‘addition’ and ‘multiplication’, defined by

\[
\forall a \forall b \forall c \forall d ((\langle a, b \rangle, \langle c, d \rangle) \in C \rightarrow (\langle a, b \rangle) + (\langle c, d \rangle) = \langle a + c, b + d \rangle)
\]

\[
\forall a \forall b \forall c \forall d ((\langle a, b \rangle, \langle c, d \rangle) \in C \rightarrow (\langle a, b \rangle) \cdot (\langle c, d \rangle) = \langle ac - bd, ad + bc \rangle)
\]

The complex numbers obey the same field axioms as the rational and real numbers with \( \langle 0,0 \rangle \) representing the 'additive identity', \( \langle 1,0 \rangle \), the 'multiplicative identity', \(-\langle a, b \rangle\), the 'additive inverse' and

\[
\left( \frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2} \right), \quad a, b \neq 0,
\]

the 'multiplicative inverse'.

Complex number addition is a valid operation within objective mathematics, since it is just vector addition. On the other hand, complex number multiplication, as ordinarily defined, is generally an invalid operation. There is no provision in objective mathematics for turning an imaginary number into a real number and vice versa i.e.

\[
\langle a, b \rangle \cdot (c, d) = (ac, bd) \neq (ac - bd, ad + bc)
\]
3.6.2.1 Number Interference

Let

\[ \sqrt{C} \equiv \{ (\pm \sqrt{x}, \pm \sqrt{y}) \}, \quad \sqrt{x}, \sqrt{y} \in \sqrt{R}, \quad x, y \geq 0 \in R \]

This implies that

\[ C = \{ (\pm \sqrt{x} \sqrt{x}, \pm \sqrt{y} \sqrt{y}) \} = \{ (\pm x, \pm y) \} \]

Now recall that

\[ \sqrt{R} = \{ \pm \sqrt{x} \}, \quad \sqrt{R} = \{ \pm i \sqrt{y} \}, \quad x, y \geq 0 \in R, \quad i = \sqrt{-1} \]

Let

\[ \left( \sqrt{R}, \sqrt{R} \right) = \{ (\pm \sqrt{x}, \pm \sqrt{y}) \} \in \sqrt{C} \rightarrow (R, \bar{R}) = \{ (\pm x, \pm y) \} \in C, \quad i^2 = -1, \]

where \( (R, \bar{R}) \) is called the set of 'complex numbers'. The rule \( i^2 = -1 \) has been restored. Note that the complex numbers are not yet instantiated.

The quantity \( (\sqrt{x}, -\sqrt{y}) \in \sqrt{C} \) is called the 'conjugate' of \( (\sqrt{x}, \sqrt{y}) \). Hence,

\[ (\sqrt{x}, -\sqrt{y}) \rightarrow (\sqrt{x} \sqrt{x}, -\sqrt{y} \sqrt{y}) = (x, -y) \in C \]

Define the 'anti-conjugate' as

\[ (-\sqrt{x}, \sqrt{y}) \in \sqrt{C} \rightarrow (-\sqrt{x} \sqrt{x}, \sqrt{y} \sqrt{y}) = (-x, y) \in C \]

Note that

\[ (x, -y) + (-x, y) = (0,0) \]

Now

\[ (\sqrt{x}, \sqrt{y}) \cdot (\sqrt{x}, -\sqrt{y}) = x + y = r \rightarrow \sqrt{r} = \pm \sqrt{x} \sqrt{y} \in \sqrt{R} \rightarrow (x, y) (x, -y) = x^2 + y^2 = r_R^2 \]

\[ \rightarrow r_R = \pm \sqrt{x^2 + y^2} \in R, \]

where ordinary complex number multiplication has been restored. On the other hand,

\[ (\sqrt{x}, \sqrt{y}) \cdot (-\sqrt{x}, \sqrt{y}) = -x + (-y) = -(x + y) = -r \rightarrow i \sqrt{r} = \pm i \sqrt{x} \sqrt{y} \in \sqrt{R} \]

\[ \rightarrow (x, y) (-x, y) = x^2 - y^2 = -(x^2 + y^2) \rightarrow ir_R = \pm i \sqrt{x^2 + y^2} \in \bar{R} \]

Let \( \sqrt{y} \) be instantiated by \( -\sqrt{y} \). In other words,

\[ \forall \sqrt{y} \exists f (\sqrt{y} \in I_m(\sqrt{C}) \rightarrow f(\sqrt{y}) = -\sqrt{y}), \quad y \geq 0 \in R \]
Note that $\sqrt{r}$ instantiates $\sqrt{y}$, but not $\sqrt{x}$. To instantiate $\sqrt{x}$, use the anti-conjugate:

$$\langle -\sqrt{x}, \sqrt{y} \rangle \rightarrow \forall \sqrt{x} \exists f (\sqrt{x} \in R_e \rightarrow f(\sqrt{x}) = -\sqrt{x}), \quad x \geq 0 \in R$$

Hence, $i\sqrt{r}$ instantiates $\sqrt{x}$, but not $\sqrt{y}$ (see fig. 3.6.2.1-1).

![Figure 3.6.2.1-1](image)

Note that this process only instantiates numbers in the first quadrant.

3.6.2.1.1 Instantiating the Complex Numbers

Consider

$$\langle \pm \sqrt{x}, \pm \sqrt{y} \rangle \sqrt{r} = \langle \pm \sqrt{x}, \pm \sqrt{y} \rangle \sqrt{x + y}, \quad r = x + y, \quad x, y \geq 0 \in R$$

Since $\sqrt{x + y}$ instantiates $\sqrt{y}$, but not $\sqrt{x}$ and $i\sqrt{x + y}$ instantiates $\sqrt{x}$, but not $\sqrt{y}$, then

$$R_e(\pm \sqrt{x}, \pm \sqrt{y}) = \pm \sqrt{x} \overset{f}{\rightarrow} \pm i \sqrt{r} = \pm i \sqrt{x + y}$$

$$I_m(\pm \sqrt{x}, \pm \sqrt{y}) = \pm i \sqrt{r} \overset{f}{\rightarrow} \pm \sqrt{x} = \pm \sqrt{x + y}$$

Let $x \rightarrow y$, $y \rightarrow x$, then

$$\pm i \sqrt{y} \overset{f}{\rightarrow} \pm i \sqrt{r} = \pm i \sqrt{x + y}, \quad \pm \sqrt{x} \overset{f}{\rightarrow} \pm \sqrt{r} = \pm \sqrt{x + y}$$

Hence,

$$R_e(\pm i \sqrt{x}, \pm i \sqrt{y}) = \pm x \overset{f}{\rightarrow} \pm i \sqrt{r} = \pm i \sqrt{x + y}$$

$$I_m(\pm i \sqrt{x}, \pm i \sqrt{y}) = \pm i y \overset{f}{\rightarrow} \pm i \sqrt{r} = \pm i \sqrt{y \sqrt{x + y}}$$

Therefore,

$$\langle \pm x, \pm y \rangle \overset{f}{\rightarrow} \langle \pm i \sqrt{x + y}, \pm \sqrt{y \sqrt{x + y}} \rangle, \quad x, y \geq 0 \in R$$
3.6.2.1.2 Instantiation Space

The complex numbers ‘\((\pm x, \pm y)\)’ are instantiated by \((\pm \sqrt{x}\sqrt{x+y}, \pm \sqrt{y}\sqrt{x+y})\) i.e.

\[ (\pm x, \pm y) \xrightarrow{f} (\pm \bar{x}, \pm \bar{y}), \quad \bar{x} = \sqrt{x}\sqrt{x+y}, \quad \bar{y} = \sqrt{y}\sqrt{x+y}, \quad x, y \geq 0 \in \mathbb{R}, \]

where \((x, y) \in C\) and \((\sqrt{x}\sqrt{x+y}, \sqrt{y}\sqrt{x+y}) \in \tilde{C}\). Note that \(C \xrightarrow{f} \tilde{C}\) is a valid system (see fig. 3.6.2.1-1),

\[ C \quad \tilde{C} \]

\( f(x, y) \rightarrow (\bar{x}, \bar{y}) \)

\[ \sqrt{x}\sqrt{x+y}, \sqrt{y}\sqrt{x+y} \]

Figure 3.6.2.1-1

where \(\tilde{C}\) is called ‘instantiation’ space. Moreover, instantiations are not unique. If

\[ (\sqrt{x}\sqrt{x+y}, \sqrt{y}\sqrt{x+y}) \xrightarrow{g} (\sqrt{y}\sqrt{x+y}, -\sqrt{x}\sqrt{x+y}), \]

where the right-hand side is obtained by multiplying \((\sqrt{x}\sqrt{x+y}, \sqrt{y}\sqrt{x+y})\) by \(-i\), then

\[ (\sqrt{x}\sqrt{x+y}, \sqrt{y}\sqrt{x+y}) \xrightarrow{g} (\sqrt{y}\sqrt{x+y}, -\sqrt{x}\sqrt{x+y}) \]

is also a valid system. In other words,

\[ (x, y) \xrightarrow{g \circ f} (\sqrt{y}\sqrt{x+y}, -\sqrt{x}\sqrt{x+y}) \]

is valid. Note that if

\[ (\sqrt{x}\sqrt{x+y}, \sqrt{y}\sqrt{x+y}) \in \tilde{C}, \]

then the real and imaginary parts of the complex number are not independent. If \(x \neq 0\), an exact value for \(x\) can only be obtained if \(y = 0\) i.e.

\[ \sqrt{x}\sqrt{x+y} = \sqrt{x}\sqrt{x} = x, \quad y = 0 \]

Conversely, an exact value for \(y \neq 0\) is obtained only if \(x = 0\).

This phenomenon is called ‘number interference’ or ‘number entanglement’. Note that if \(x\) and \(y\) have dimension ‘\(d\)’, then
\[ \sqrt{x + y} = \sqrt{x} + \sqrt{y} \]

In other words, \( \sqrt{x + y} \) comes in whole units.

### 3.7 Anomalies within Instantiation Space

In the previous section, it was assumed that \( x, y \geq 0 \in R \). If this restriction is eliminated, then some anomalies arise. If \( -x, -y < 0 \), then \( x, y > 0 \) and

\[
\langle x, y \rangle \xrightarrow{f} \langle \sqrt{x} + y, \sqrt{x + y} \rangle \in C,
\]

\[
\langle -x, -y \rangle \xrightarrow{f} \langle \sqrt{-x} - y, \sqrt{-x + y} \rangle = \langle iy, \sqrt{x} + yi \rangle,
\]

\[
\langle -x, y \rangle \xrightarrow{f} \langle \sqrt{-x} + y, \sqrt{x + y} \rangle = \langle iy, \sqrt{x} + yi \rangle.
\]

However, if \( x, y \) have different signs, then, depending on whether \( x > y \) or \( y > x \), either

\[ \sqrt{x} + y, \quad \sqrt{y} + x, \]

but not both, will be imaginary. To see this, suppose \( x > 0, -y < 0 \). If \( x - y > 0 \), then

\[ \sqrt{x - y} \in R, \quad \sqrt{-y} \in R \rightarrow \langle \sqrt{x - y}, \sqrt{-y} \rangle \notin C \]

If \( x - y < 0 \), then

\[ \sqrt{x - y} \notin R, \quad \sqrt{-y} \notin R \rightarrow \langle \sqrt{x - y}, \sqrt{-y} \rangle \notin C \]

This is a common feature of complex numbers, since

\[ \sqrt{-y} \neq \sqrt{(-x)(-y)}, \quad x, y \geq 0 \in R \]

The instantiation of complex numbers requires that \( x, y \geq 0 \in R \). This restriction can be removed in a couple of ways. One way is to let

\[
\begin{align*}
\langle x, y \rangle & \xrightarrow{f} \langle \sqrt{x} + y, \sqrt{y} \rangle, \\
\langle -x, -y \rangle & \xrightarrow{f} \langle -\sqrt{y} \sqrt{x}, \sqrt{x} + y \rangle, \\
\langle -x, +y \rangle & \xrightarrow{-f} \langle -\sqrt{x} \sqrt{y}, -\sqrt{y} \rangle, \\
\langle +x, -y \rangle & \xrightarrow{-f} \langle \sqrt{x} \sqrt{y}, -\sqrt{x} \rangle.
\end{align*}
\]

On the other hand, let

\[
f(x) = \begin{cases} 
\sqrt{|x| + |y|}, & \text{if } x \geq 0 \\
-\sqrt{|x| + |y|}, & \text{if } x < 0
\end{cases}, \quad f(y) = \begin{cases} 
\sqrt{|y| + |x|}, & \text{if } y \geq 0 \\
-\sqrt{|y| + |x|}, & \text{if } y < 0
\end{cases}
\]

56
which is simply a formal statement of the approach to instantiation previously undertaken. However, the two approaches described above produce different instantiations.

### 3.7.1 Operations on Complex Numbers in Objective Space

Operations on complex numbers in objective space are listed below along with their instantiations.

1. **Addition:**

\[
\langle a, b \rangle + \langle c, d \rangle = \langle a + c, b + d \rangle \rightarrow \left\langle (\sqrt{(a + c)}'), (\sqrt{(b + d)})' \right\rangle \sqrt{\left(\sqrt{(a + c)}'\right)^2 + \left(\sqrt{(b + d)}'\right)^2} = \left\langle \sqrt{(a + c)}', \sqrt{(b + d)}' \right\rangle \sqrt{|a + c| + |b + d|}
\]

2. **Multiplication:**

\[
\langle a, b \rangle \cdot \langle c, d \rangle = \langle ac - bd, ad + bc \rangle \rightarrow \left\langle (\sqrt{(ac - bd)})', (\sqrt{(ad + bc)})' \right\rangle \sqrt{\left(\sqrt{(ac - bd)}'\right)^2 + \left(\sqrt{(ad + bc)}'\right)^2} = \left\langle \sqrt{(ac - bd)}', \sqrt{(ad + bc)}' \right\rangle \sqrt{|ac - bd| + |ad + bc|}
\]

3. **Additive Identity:** \(\langle 0, 0 \rangle \rightarrow \langle \sqrt{0}, \sqrt{0} \rangle \sqrt{0} = \langle 0, 0 \rangle\)

4. **Additive Inverse:** \(\langle -a, -b \rangle \rightarrow \langle -\sqrt{a}^{'}, -\sqrt{b}^{'}, \sqrt{|a|} + |b| \rangle
\]

5. **Multiplicative Identity:** \(\langle 1, 0 \rangle \rightarrow \langle \sqrt{1}, \sqrt{0} \rangle \sqrt{1} = \langle 1, 0 \rangle\)

6. **Multiplicative Inverse:**

\[
\left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}\right) \rightarrow \left\langle \frac{\sqrt{a}'}{(|a| + |b|)^2}, \frac{-\sqrt{b}'}{(|a| + |b|)^2} \right\rangle \sqrt{|a| + |b|}, \quad a, b \neq 0
\]

Note that \(i^2 = -1\).

---

1 The rules are valid if and only if

\[
x' = \begin{cases} 
\sqrt{|a|}, & \text{if } a \geq 0 \\
-\sqrt{|a|}, & \text{if } a < 0
\end{cases}, \quad \quad y' = \begin{cases} 
\sqrt{|b|}, & \text{if } b \geq 0 \\
-\sqrt{|b|}, & \text{if } b < 0
\end{cases}, \quad \quad a, b \in R
\]
3.8 Concluding Remarks

Objective mathematics requires that the natural, integer, rational and real number systems be instantiated. Instantiated number systems are similar, but not identical, to ordinary number systems. In fact, some operations, valid in ordinary mathematics, are invalid within objective mathematics.

The idea of ‘instantiation’ leads immediately to the complex plane, where numbers lying along the real axis are instantiated by numbers lying along the imaginary axis. Moreover, the operations associated with a vector space, ‘vector addition’ and ‘scalar multiplication’, are valid within objective mathematics. However, complex number multiplication is incompatible with objective mathematics. For instance, the metric (distance), represented by an inner product of two vectors, has no instantiation in objective mathematics. The metric within objective mathematics is defined

$$\sqrt{r} = \sqrt{x + y},$$

which is not the same as the inner product in a Hilbert space.

Instantiation of a complex number is defined as

$$\forall x \forall y \exists f(x, y \in \mathbb{C} \rightarrow f((\pm x, \pm y)) = \langle \bar{x}, \bar{y} \rangle \in \bar{\mathbb{C}}, \quad \bar{x} = \pm \sqrt{x} \sqrt{x + y}, \quad \bar{y} = \pm \sqrt{y} \sqrt{x + y}),$$

$$x, y \geq 0$$

The instantiation requirement within objective mathematics leads to ‘number interference’, where the imaginary part of an instantiated complex number interferes with the real part and vice versa and exact values for instantiated numbers can be obtained only if one of the parts, real or imaginary, is zero.
Chapter 4

Complex Analysis in Instantiation Space

“I always like to look on the optimistic side of life, but I am realistic enough to know that life is a complex matter.”

~ Walt Disney

4.0 Introduction

In Chapter 3, objective number systems were developed, including objective complex numbers. The instantiation of the complex numbers resulted in ‘number interference’, where the real part of a complex number interferes with the imaginary part and vice versa. Number interference arises because complex number multiplication is incompatible with objective mathematics.

Complex numbers in instantiation space can be represented by the complex pair

\[ \langle \sqrt{a\sqrt{a + b}}, \sqrt{b\sqrt{a + b}} \rangle, \quad a, b \geq 0, \quad a, b < 0, \]

or by

\[ \langle -\sqrt{|a|\sqrt{|a| + |b|}}, \sqrt{|b|\sqrt{|a| + |b|}} \rangle, \quad -a, b \geq 0, \quad \langle \sqrt{|a|\sqrt{|a| + |b|}}, -\sqrt{|b|\sqrt{|a| + |b|}} \rangle, \]

\[ a, -b \geq 0 \]

However, complex numbers in instantiation space suffer anomalies which do not arise when working with ordinary complex numbers. This chapter will examine the concept of ‘number interference’ and the anomalies that arise from it.

4.1 Instantiated Complex Numbers

In instantiation space, a complex number ‘\( \bar{z} \)’ is represented by

\[ \bar{z} = \langle \sqrt{x}, \sqrt{y} \rangle \sqrt{x + y} \in \tilde{C}, \]

which can be written

\[ \forall x, y \left( z = \langle x, y \rangle \in C \rightarrow \bar{z} = \langle \tilde{x}, \tilde{y} \rangle \in \tilde{C}, \quad \tilde{x} = \sqrt{x} \sqrt{x + y}, \quad \tilde{y} = \sqrt{y} \sqrt{x + y} \right), \]

where \( \tilde{x} \) instantiates \( x \) and \( \tilde{y} \) instantiates \( y \).

4.1.1 Discontinuities in Instantiation Space

Recall that \( x, y \) can be written in polar coordinates:

\[ x = r \cos \theta, \quad y = r \sin \theta, \quad r = \sqrt{x^2 + y^2} \]
Note that
\[ x = r \cos \theta \quad \sqrt{x} = \sqrt{r} \cos \theta \]
\[ y = r \sin \theta \quad \sqrt{y} = \sqrt{r} \sin \theta \]

The square roots create anomalies when \( x < 0 \) or \( y < 0 \). For instance, if \( 0 \leq \theta < 2\pi \), then if
\[ \theta = \frac{7\pi}{4} \rightarrow x = \frac{\sqrt{2}}{2} r, \quad y = -\frac{\sqrt{2}}{2} r, \quad \sqrt{x} = \sqrt{\frac{\sqrt{2}}{2}}, \quad \sqrt{y} = \sqrt{-\frac{\sqrt{2}}{2}} \]

Projection of \( \langle x, y \rangle \) into instantiation space gives
\[ \langle x, y \rangle \rightarrow \frac{\sqrt{2}}{2} \langle \sqrt{r}, \sqrt{-r} \rangle \sqrt{r} - r = (0, 0), \quad (0, 0) \rightarrow (\sqrt{0}, \sqrt{0}) \sqrt{0} + 0 = (0, 0), \quad r = \sqrt{x^2 + y^2} \]

Therefore, more than one distinct nominal number maps to the same instantiation, an inconsistency. To ensure a valid system, each complex pair \( \langle x, y \rangle \) must have a unique instantiation.

The 'polar' form of a complex number \( \bar{z} \) in instantiation space is written
\[ \bar{z} = \sqrt{r} \left( \cos^{1/2} \theta + i \sin^{1/2} \theta \right) \sqrt{r} \cos \theta + r \sin \theta = r \left( \cos^{1/2} \theta + i \sin^{1/2} \theta \right) \sqrt{\cos \theta + \sin \theta} \]
\( r \geq 0 \)

The conjugate \( \bar{z}^* \) of an instantiated complex number \( \bar{z} \), written in polar form, is
\[ \bar{z}^* = r \left( \cos^{1/2} \theta - i \sin^{1/2} \theta \right) \sqrt{\cos \theta + \sin \theta} \rightarrow \bar{z} \bar{z}^* = r^2 (\cos \theta + \sin \theta)^2 \]
\[ = (r \cos \theta + r \sin \theta)^2 = (x + y)^2 \rightarrow |\bar{z} \bar{z}^*| = |x + y|^2 = |x + y| \]
\[ \rightarrow +\sqrt{|x + y|} \geq 0 \]

Similar to ordinary polar arithmetic, where the modulus is positive, the metric \( \sqrt{x^2 + y^2} \) within instantiation space should be positive or zero.

Consider
\[ [R_e(\bar{z})]^2 = r^2 \cos \theta (\cos \theta + \sin \theta) = r^2 (\cos^2 \theta + \cos \theta \sin \theta) \]
\[ = r^2 \left( \frac{1 + \cos(2\theta) + \sin(2\theta)}{2} \right), \quad r \geq 0 \]

By the same token,
\[ [I_m(\bar{z})]^2 = r^2 \left( \frac{1 - \cos(2\theta) + \sin(2\theta)}{2} \right) \]

However, for certain values of \( \theta \), \( [R_e(\bar{z})]^2 < 0 \). The same can be said of \( [I_m(\bar{z})]^2 \). Both the imaginary and real part of the complex number can be made positive if
\[ |[\mathcal{R}(\bar{z})]|^2 = r^2 \frac{|1 + \cos(2\theta) + \sin(2\theta)|}{2}, \quad |[\mathcal{I}(\bar{z})]|^2 = r^2 \frac{|1 - \cos(2\theta) + \sin(2\theta)|}{2} \]

Taking the positive square roots of \(|[\mathcal{R}(\bar{z})]|^2| \) and \(|[\mathcal{I}(\bar{z})]|^2| \) respectfully,

\[ |\mathcal{R}(\bar{z})| = \frac{\sqrt{2}r}{2} \sqrt{|1 + \cos(2\theta) + \sin(2\theta)|} \in \{R_+, 0\}, \quad |\mathcal{I}(\bar{z})| = \frac{\sqrt{2}r}{2} \sqrt{|1 - \cos(2\theta) + \sin(2\theta)|} \in \{R_+, 0\} \]

Define

\[ |\bar{z}| = \frac{\sqrt{2}r}{2} \left( \sqrt{|1 + \cos(2\theta) + \sin(2\theta)|} + i\sqrt{|1 - \cos(2\theta) + \sin(2\theta)|} \right), \quad 0 \leq \theta < \pi \]

The locus of

\[ \langle |[\mathcal{R}(\bar{z})]|^2, |[\mathcal{I}(\bar{z})]|^2 \rangle = \left( \frac{1 + \cos(2\theta) + \sin(2\theta)}{2}, \frac{1 - \cos(2\theta) + \sin(2\theta)}{2} \right), \quad r = 1 \]

is shown in fig. 4.1.1-1. It is a complex circle of radius ‘\(\sqrt{2}/2\)’ with center ‘\((1/2, 1/2)\)’.

![Figure 4.1.1-1](image)

The locus of

\[ \langle |[\mathcal{R}(\bar{z})]|^2, |[\mathcal{I}(\bar{z})]|^2 \rangle = \left( \frac{1 + \cos(2\theta) + \sin(2\theta)}{2}, \frac{1 - \cos(2\theta) + \sin(2\theta)}{2} \right), \quad r = 1 \]

is shown in fig. 4.1.1-2.
Discontinuities arise at the quadrant boundaries as a consequence of inserting absolute values for the real and imaginary parts of the complex number. The same discontinuities arise if the positive square roots are used instead (see fig. 4.1.1-3):

\[
\langle |R_e(\ddot{z})|, |I_m(\ddot{z})| \rangle = \left( \sqrt{\frac{1 + \cos(2\theta) + \sin(2\theta)}{2}}, \sqrt{\frac{1 - \cos(2\theta) + \sin(2\theta)}{2}} \right), \quad r = 1
\]

Again, the discontinuities appear at the quadrant boundaries. Let

\[
\bar{x}' = \frac{1 + \cos(2\theta) + \sin(2\theta)}{2}, \quad \bar{y}' = \frac{1 - \cos(2\theta) + \sin(2\theta)}{2},
\]

then the discontinuities can be overcome by recognizing that

\[
\langle \bar{x}', \bar{y}' \rangle \in \begin{cases} 
\text{First Quadrant}, & \text{if } \bar{x}' \geq 0, \quad \bar{y}' \geq 0 \\
\text{Second Quadrant}, & \text{if } \bar{x}' < 0, \quad \bar{y}' \geq 0 \\
\text{Third Quadrant}, & \text{if } \bar{x}' \leq 0, \quad \bar{y}' < 0' \\
\text{Fourth Quadrant}, & \text{if } \bar{x}' > 0, \quad \bar{y}' \leq 0
\end{cases}
\]
Hence, define

\[
\bar{z} = \begin{cases} 
(\bar{x}, \bar{y}), & \text{if } \bar{x}' \geq 0, \quad \bar{y}' \geq 0 \\
(-\bar{x}, \bar{y}), & \text{if } \bar{x}' < 0, \quad \bar{y}' \geq 0 \\
(-\bar{x}, -\bar{y}), & \text{if } \bar{x}' \leq 0, \quad \bar{y}' < 0' \\
(\bar{x}, -\bar{y}), & \text{if } \bar{x}' > 0, \quad \bar{y}' \leq 0 
\end{cases}
\]

where

\[
\bar{x} = \sqrt{\frac{1 + \cos(2\theta) + \sin(2\theta)}{2}}, \quad \bar{y} = \sqrt{\frac{1 - \cos(2\theta) + \sin(2\theta)}{2}}
\]

The locus of \(\bar{z}\) is shown in fig. 4.1.1-4.

**Figure 4.1.1-4**

### 4.1.2 Guaranteeing Unique Instantiations

Solving the equations ‘\(\sqrt{\bar{x}}\sqrt{\bar{x} + \bar{y}} = 0\)’ and ‘\(\sqrt{\bar{y}}\sqrt{\bar{x} + \bar{y}} = 0\)’ for \(\bar{y}\) yields

\[
y = -\bar{x}, \quad \bar{y} = 0
\]

The points lying along the line ‘\(\bar{y} = -\bar{x}\)’ in the complex plane are troublesome because these points do not have unique instantiations. The problem is overcome by defining

\[
\bar{z} = \begin{cases} 
(\bar{x}, \bar{y}), & \text{if } \bar{x} \geq 0, \quad \bar{y} \geq 0 \\
(\bar{x}, \bar{y}), & \text{if } \bar{x} \leq 0, \quad \bar{y} \leq 0 \\
(-|\bar{x}|, |\bar{y}|), & \text{if } \bar{x} < 0, \quad \bar{y} \geq 0' \\
(|\bar{x}|, -|\bar{y}|), & \text{if } \bar{x} \geq 0, \quad \bar{y} < 0
\end{cases}
\]

where

\[
\bar{x} = \sqrt{\bar{x}}\sqrt{\bar{x} + \bar{y}}, \quad \bar{y} = \sqrt{\bar{y}}\sqrt{\bar{x} + \bar{y}}, \quad |\bar{x}| = \sqrt{|x|\sqrt{|x| + |y|}}, \quad |\bar{y}| = \sqrt{|y|\sqrt{|x| + |y|}},
\]

which guarantees a unique instantiation for each complex number ‘\((x, y)\)’.
4.2 Projection

The process of instantiating a complex number \( z = (x, y) \) is called ‘projection’ into instantiation space. In other words,

\[
f_z \to \bar{z},
\]

where the complex number \( z \in C \) is represented in instantiation space by the complex number \( \bar{z} \in \overline{C} \).

4.2.1 The Modulus of \( z \) and Arg \( z \)

A complex number \( z \) can be represented in polar form:

\[
x = r \cos \theta, \quad y = r \sin \theta, \quad r = \sqrt{x^2 + y^2}
\]

The quantity \( r = \sqrt{x^2 + y^2} \) is called the ‘modulus’ of \( z \) and \( \theta \), the ‘argument’ of \( z \), denoted ‘arg \( z \)’. However, if \( z \neq 0 \), arg \( z \) is not unique. To see this, note that

\[
z = x + iy = r(\cos \theta + i \sin \theta) = r(\cos(\theta \pm 2\pi k) + i \sin(\theta \pm 2\pi k)), \quad k = 0, 1, 2, ..., n - 1,
\]

where, for each \( k \), \( z \) represents the same point in the complex plane. To make \( z \) unique, arg \( z \) must be restricted to intervals of \( 2\pi \), for instance, \( 0 \leq \theta < 2\pi \). A particular choice for a ‘\( 2\pi \)-interval’ is called the ‘principle range’ of arg \( z \) and \( \theta \) is called the ‘principle value’.

If \( \bar{z} \) signifies \( z \) projected into instantiation space, then

\[
\bar{z} = \begin{cases} 
(\bar{x}, \bar{y}), & \text{if } \bar{x}' \geq 0, \quad \bar{y}' \geq 0 \\
(-\bar{x}, \bar{y}), & \text{if } \bar{x}' < 0, \quad \bar{y}' \geq 0 \\
(-\bar{x}, -\bar{y}), & \text{if } \bar{x}' \leq 0, \quad \bar{y}' < 0' \\
(\bar{x}, -\bar{y}), & \text{if } \bar{x}' > 0, \quad \bar{y}' \leq 0 
\end{cases}
\]

where

\[
\bar{x}' = 1 + \cos[2(\theta \pm \pi k)] + \sin[2(\theta \pm \pi k)], \quad \bar{y}' = 1 - \cos[2(\theta \pm \pi k)] + \sin[2(\theta \pm \pi k)],
\]

\[
\bar{x} = \frac{\sqrt{2}r}{2} \sqrt{1 + \cos[2(\theta \pm \pi k)] + \sin[2(\theta \pm \pi k)]}, \\
\bar{y} = \frac{\sqrt{2}r}{2} \sqrt{1 - \cos[2(\theta \pm \pi k)] + \sin[2(\theta \pm \pi k)]}, \quad k = 0, 1, 2, ..., n - 1, \\
0 \leq \theta < \pi
\]

In order to maintain a unique instantiation, arg \( z \) must be confined to its principle range.
4.2.2 The Product of Complex Numbers in Instantiation Space

Suppose

\[ z_1 = r_1 (\cos \theta_1 + i \sin \theta_1) = r_1 e^{i \theta_1}, \quad z_2 = r_2 (\cos \theta_2 + i \sin \theta_2) = r_2 e^{i \theta_2}, \]

then

\[ z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)} = r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)] \rightarrow \overline{z_1 z_2} \]

\[ = \begin{cases} 
(\bar{x}, \bar{y}), & \text{if } \bar{x}' \geq 0, \quad \bar{y}' \geq 0 \\
(\bar{x}, \bar{y}), & \text{if } \bar{x}' < 0, \quad \bar{y}' \geq 0 \\
(-\bar{x}, \bar{y}), & \text{if } \bar{x}' \leq 0, \quad \bar{y}' < 0' \\
(\bar{x}, -\bar{y}), & \text{if } \bar{x}' > 0, \quad \bar{y}' \leq 0
\end{cases} \]

where

\[ \bar{x}' = 1 + \cos[2(\theta_1 + \theta_2)] + \sin[2(\theta_1 + \theta_2)], \]
\[ \bar{y}' = 1 - \cos[2(\theta_1 + \theta_2)] + \sin[2(\theta_1 + \theta_2)], \]
\[ \bar{x} = \frac{\sqrt{2} r_1 r_2}{2} \sqrt{1 + \cos[2(\theta_1 + \theta_2)] + \sin[2(\theta_1 + \theta_2)]}, \]
\[ \bar{y} = \frac{\sqrt{2} r_1 r_2}{2} \sqrt{1 - \cos[2(\theta_1 + \theta_2)] + \sin[2(\theta_1 + \theta_2)]}, \]

and

\[ \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)} = \frac{r_1}{r_2} (\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)) \rightarrow \frac{\bar{z}_1}{\bar{z}_2} = \]

\[ = \begin{cases} 
(\bar{x}, \bar{y}), & \text{if } \bar{x}' \geq 0, \quad \bar{y}' \geq 0 \\
(-\bar{x}, \bar{y}), & \text{if } \bar{x}' < 0, \quad \bar{y}' \geq 0 \\
(-\bar{x}, -\bar{y}), & \text{if } \bar{x}' \leq 0, \quad \bar{y}' < 0' \\
(\bar{x}, -\bar{y}), & \text{if } \bar{x}' > 0, \quad \bar{y}' \leq 0
\end{cases} \]

where

\[ \bar{x}' = 1 + \cos[2(\theta_1 - \theta_2)] + \sin[2(\theta_1 - \theta_2)], \]
\[ \bar{y}' = 1 - \cos[2(\theta_1 - \theta_2)] + \sin[2(\theta_1 - \theta_2)], \]
\[ \bar{x} = \frac{\sqrt{2} r_1}{2 r_2} \sqrt{1 + \cos[2(\theta_1 - \theta_2)] + \sin[2(\theta_1 - \theta_2)]}, \]
\[ \bar{y} = \frac{\sqrt{2} r_1}{2 r_2} \sqrt{1 - \cos[2(\theta_1 - \theta_2)] + \sin[2(\theta_1 - \theta_2)]}. \]
4.2.2.1 De Moivre’s Theorem Instantiated

In general,
\[ z_1z_2...z_n = r_1r_2...r_n[\cos(\theta_1 + \theta_2 + ⋯ + \theta_n) + i\sin(\theta_1 + \theta_2 + ⋯ + \theta_n)] \rightarrow z_1z_2...z_n \]

\[ \begin{cases} (\bar{x}, \bar{y}), & \text{if } \bar{x}' \geq 0, \quad \bar{y}' \geq 0 \\ (-\bar{x}, \bar{y}), & \text{if } \bar{x}' < 0, \quad \bar{y}' \geq 0 \\ (-\bar{x}, -\bar{y}), & \text{if } \bar{x}' \leq 0, \quad \bar{y}' < 0' \\ (\bar{x}, -\bar{y}), & \text{if } \bar{x}' > 0, \quad \bar{y}' \leq 0' \end{cases} \]

where
\[ \bar{x}' = 1 + \cos[2(\theta_1 + \theta_2 + ⋯ + \theta_n)] + \sin[2(\theta_1 + \theta_2 + ⋯ + \theta_n)], \]
\[ \bar{y}' = 1 - \cos[2(\theta_1 + \theta_2 + ⋯ + \theta_n)] + \sin[2(\theta_1 + \theta_2 + ⋯ + \theta_n)], \]
\[ \bar{x} = \frac{\sqrt{2}}{2} r_1r_2...r_n\sqrt{[1 + \cos[2(\theta_1 + \theta_2 + ⋯ + \theta_n)] + \sin[2(\theta_1 + \theta_2 + ⋯ + \theta_n)]]}, \]
\[ \bar{y} = \frac{\sqrt{2}}{2} r_1r_2...r_n\sqrt{[1 - \cos[2(\theta_1 + \theta_2 + ⋯ + \theta_n)] + \sin[2(\theta_1 + \theta_2 + ⋯ + \theta_n)]]} \]

If \( z_1 = z_2 = ⋯ = z_n = z \), then
\[ z^n = [r(\cos \theta + i \sin \theta)]^n = r^n[\cos(n\theta) + i \sin(n\theta)] \rightarrow \bar{z^n} \]

\[ \begin{cases} (\bar{x}, \bar{y}), & \text{if } \bar{x}' \geq 0, \quad \bar{y}' \geq 0 \\ (-\bar{x}, \bar{y}), & \text{if } \bar{x}' < 0, \quad \bar{y}' \geq 0 \\ (-\bar{x}, -\bar{y}), & \text{if } \bar{x}' \leq 0, \quad \bar{y}' < 0' \\ (\bar{x}, -\bar{y}), & \text{if } \bar{x}' > 0, \quad \bar{y}' \leq 0' \end{cases} \]

where
\[ \bar{x}' = 1 + \cos(2n\theta) + \sin(2n\theta), \]
\[ \bar{y}' = 1 - \cos(2n\theta) + \sin(2n\theta), \]
\[ \bar{x} = \frac{\sqrt{2}}{2} r^n\sqrt{[1 + \cos(2n\theta) + \sin(2n\theta)]}, \]
\[ \bar{y} = \frac{\sqrt{2}}{2} r^n\sqrt{[1 - \cos(2n\theta) + \sin(2n\theta)]} \]

which is De Moivre’s theorem instantiated.

4.2.3 The \( n \)th Roots of \( z \) Instantiated

In complex space, the number \('z^{1/n}'\) is called the \('n^{th}\) root’ of \( z \). By De Moivre’s theorem, if \( n > 0 \in N \) and \( z \neq 0 \), replacing \( n \) with \( 1/n \) results in
\[ z^{1/n} = [r(\cos \theta + i \sin \theta)]^{1/n} = r^{1/n} \left[ \cos \left( \frac{\theta}{n} \right) + i \sin \left( \frac{\theta}{n} \right) \right] \rightarrow z^{1/n} \]

\[
= \begin{cases} 
(\bar{x}, \bar{y}), & \text{if } \bar{x}' \geq 0, \quad \bar{y}' \geq 0 \\
(-\bar{x}, \bar{y}), & \text{if } \bar{x}' < 0, \quad \bar{y}' \geq 0 \\
(-\bar{x}, -\bar{y}), & \text{if } \bar{x}' \leq 0, \quad \bar{y}' < 0' \\
(\bar{x}, -\bar{y}), & \text{if } \bar{x}' > 0, \quad \bar{y}' \leq 0
\end{cases}
\]

where

\[
\bar{x}' = 1 + \cos \left( \frac{2\theta}{n} \right) + \sin \left( \frac{2\theta}{n} \right),
\]

\[
\bar{y}' = 1 - \cos \left( \frac{2\theta}{n} \right) + \sin \left( \frac{2\theta}{n} \right),
\]

\[
\bar{x} = \frac{\sqrt{2}}{2} r^{1/n} \sqrt{1 + \cos \left( \frac{2\theta}{n} \right) + \sin \left( \frac{2\theta}{n} \right)},
\]

\[
\bar{y} = \frac{\sqrt{2}}{2} r^{1/n} \sqrt{1 - \cos \left( \frac{2\theta}{n} \right) + \sin \left( \frac{2\theta}{n} \right)},
\]

where \( z^{1/n} \) is the \( n \)th root of \( z \) instantiated. In general,

\[
z^{1/n} = r^{1/n} \left[ \cos \left( \frac{\theta + 2\pi k}{n} \right) + i \sin \left( \frac{\theta + 2\pi k}{n} \right) \right],
\]

\[
k = 0, 1, 2, \ldots, n - 1 \rightarrow \bar{z}^{1/n} = \begin{cases} 
(\bar{x}, \bar{y}), & \text{if } \bar{x}' \geq 0, \quad \bar{y}' \geq 0 \\
(-\bar{x}, \bar{y}), & \text{if } \bar{x}' < 0, \quad \bar{y}' \geq 0 \\
(-\bar{x}, -\bar{y}), & \text{if } \bar{x}' \leq 0, \quad \bar{y}' < 0' \\
(\bar{x}, -\bar{y}), & \text{if } \bar{x}' > 0, \quad \bar{y}' \leq 0
\end{cases}
\]

where

\[
\bar{x}' = 1 + \cos \left[ \frac{2(\theta + \pi k)}{n} \right] + \sin \left[ \frac{2(\theta + \pi k)}{n} \right],
\]

\[
\bar{y}' = 1 - \cos \left[ \frac{2(\theta + \pi k)}{n} \right] + \sin \left[ \frac{2(\theta + \pi k)}{n} \right],
\]

\[
\bar{x} = \frac{\sqrt{2}}{2} r^{1/n} \sqrt{1 + \cos \left[ \frac{2(\theta + \pi k)}{n} \right] + \sin \left[ \frac{2(\theta + \pi k)}{n} \right]},
\]

\[
\bar{y} = \frac{\sqrt{2}}{2} r^{1/n} \sqrt{1 - \cos \left[ \frac{2(\theta + \pi k)}{n} \right] + \sin \left[ \frac{2(\theta + \pi k)}{n} \right]},
\]
where each $k$ is said to be on a different ‘branch’.

There are $n$ roots of $z$. Importantly, if
\[ a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0, \]
where the $a_i$’s are complex numbers and $n \in \{N, 0\}$, then, by the fundamental theorem of algebra, the above polynomial has $n$ roots, not necessarily distinct. Hence,
\[ a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0, \quad a_0 \neq 0 \]
can be factored into
\[ a_0(z - z_1)(z - z_2) \cdots (z - z_n) = 0 \]
The $z_i$’s are the $n$ roots of the equation.

### 4.2.3.1 The $n^{\text{th}}$ Roots of Unity Instantiated

The roots of the equation $z^n = 1$, $n \in \mathbb{N}$ are called the ‘$n^{\text{th}}$ roots of unity’ given by
\[ z = \cos \left( \frac{2\pi k}{n} \right) + i \sin \left( \frac{2\pi k}{n} \right) = e^{i\frac{2\pi k}{n}}, \quad k = 0, 1, 2, \ldots, n - 1 \]

Let
\[ \omega_k = \cos \left( \frac{2\pi k}{n} \right) + i \sin \left( \frac{2\pi k}{n} \right) = e^{i\frac{2\pi k}{n}}, \]
then
\[ \vec{\omega}_k = \begin{cases} (\bar{x}, \bar{y}), & \text{if } \bar{x}' \geq 0, \quad \bar{y}' \geq 0 \\ (-\bar{x}, \bar{y}), & \text{if } \bar{x}' < 0, \quad \bar{y}' \geq 0 \\ (-\bar{x}, -\bar{y}), & \text{if } \bar{x}' \leq 0, \quad \bar{y}' < 0 \\ (\bar{x}, -\bar{y}), & \text{if } \bar{x}' > 0, \quad \bar{y}' \leq 0 \end{cases} \]

where
\[ \bar{x}' = 1 + \cos \left( \frac{2\pi k}{n} \right) + \sin \left( \frac{2\pi k}{n} \right), \]
\[ \bar{y}' = 1 - \cos \left( \frac{2\pi k}{n} \right) + \sin \left( \frac{2\pi k}{n} \right), \]
\[ \bar{x} = \frac{\sqrt{2}}{2} \sqrt{1 + \cos \left( \frac{2\pi k}{n} \right) + \sin \left( \frac{2\pi k}{n} \right)}, \]
\[ \bar{y} = \frac{\sqrt{2}}{2} \sqrt{1 - \cos \left( \frac{2\pi k}{n} \right) + \sin \left( \frac{2\pi k}{n} \right)} \]
The \( n \) instantiated roots are represented by \( \bar{\omega} \).

Example: Suppose \( n = 4 \), then
\[
\begin{align*}
\omega_0 &= 1, \quad \omega_1 = i, \quad \omega_2 = -1, \quad \omega_3 = -i \\
\bar{\omega}_0 &= 1, \quad \bar{\omega}_1 = 1 + i, \quad \bar{\omega}_2 = i, \quad \bar{\omega}_3 = 0
\end{align*}
\]

4.2.4 The Dot Product Instantiated

A complex number can be thought of as a vector i.e.
\[
\forall x \forall y ((0,0), (x, y) \in C \rightarrow (x, 0) \perp (0, y)),
\]
where the point \( P = (x, y) \) sits in the complex plane and has an initial point at the origin \( '(0,0)' \) and a terminal point at \( P \). If \( z_1 = (x_1, y_1) \) and \( z_2 = (x_2, y_2) \), then
\[
z_1 + z_2 = (x_1 + x_2, y_1 + y_2),
\]
which is synonymous with vector addition. The complex number \( 'z = (x_1 + x_2, y_1 + y_2)' \) projected into instantiation space is
\[
\bar{z} = \begin{cases} 
(\bar{x}, \bar{y}), & \text{if } x_1 + x_2 \geq 0, \quad y_1 + y_2 \geq 0 \\
(\bar{x}, \bar{y}), & \text{if } x_1 + x_2 \leq 0, \quad y_1 + y_2 \leq 0 \\
(-|\bar{x}|, |\bar{y}|), & \text{if } x_1 + x_2 < 0, \quad y_1 + y_2 \geq 0' \\
(|\bar{x}|, -|\bar{y}|), & \text{if } x_1 + x_2 \geq 0, \quad y_1 + y_2 < 0
\end{cases}
\]

where
\[
\bar{x} = \sqrt{x_1 + x_2} \sqrt{x_1 + x_2 + y_1 + y_2}, \quad \bar{y} = \sqrt{y_1 + y_2} \sqrt{x_1 + x_2 + y_1 + y_2},
\]
\[
|\bar{x}| = \sqrt{|x_1 + x_2|} \sqrt{|x_1 + x_2| + |y_1 + y_2|},
\]
\[
|\bar{y}| = \sqrt{|y_1 + y_2|} \sqrt{|x_1 + x_2| + |y_1 + y_2|},
\]

where \( \bar{z} \) is \( z \) instantiated.

The dot product \( 'z_1 \circ z_2' \) of two complex numbers is defined
\[
z_1 \circ z_2 = |z_1||z_2| \cos \theta = x_1 x_2 + y_1 y_2 = R_e(z_1^* z_2), \quad \theta = \angle(z_1, z_2), \quad 0 \leq \theta < \pi
\]
To see this, let \( z_1 = x_1 + i y_1 \) and \( z_2 = x_2 + i y_2 \), then
\[
z_1 \circ z_2 = R_e(z_1^* z_2) = R_e((x_1 - i y_1)(x_2 + i y_2)) = R_e(x_1 x_2 + i x_1 y_2 - i x_2 y_1 + y_1 y_2) = x_1 x_2 + y_1 y_2
\]

Note that the dot product of two complex numbers returns a real number. Hence,
\[
z_1 \circ z_2 = |z_1||z_2| \cos \theta = x_1 x_2 + y_1 y_2 = R_e(z_1^* z_2) \rightarrow \bar{z_1} \circ \bar{z_2} = \left(\sqrt{x_1 x_2 + y_1 y_2}, 0\right) \sqrt{x_1 x_2 + y_1 y_2 + 0} = x_1 x_2 + y_1 y_2,
\]

69
where $\overline{z_1 \circ z_2}$ is $z_1 \circ z_2$ instantiated. Note that $\overline{z_1 \circ z_2} = z_1 \circ z_2$.

### 4.2.5 The Cross Product Instantiated

The cross product of two complex numbers '$z_1, z_2$' is defined

$$\langle z_1 | \times | z_2 \rangle = |z_1||z_2| \sin \theta = x_1y_2 - y_1x_2 = l_m(z_1^* z_2)$$

Note that

$$\langle z_1 | \times | z_2 \rangle = (0, \sqrt{x_1y_2 - y_1x_2}) \sqrt{0 + x_1y_2 - y_1x_2} = l_m(x_1y_2 - y_1x_2)$$

Again, $\langle z_1 | \times | z_2 \rangle$ is $\langle z_1 | \times | z_2 \rangle$ instantiated. Note that

$$\langle z_1 | \times | z_2 \rangle = \langle z_1 | \times | z_2 \rangle$$

Moreover,

$$z_1 \circ z_2 + i \langle z_1 | \times | z_2 \rangle = |z_1||z_2|e^{i\theta} \rightarrow |z_1||z_2|e^{i\theta} = x_1y_2 + i(x_1y_2 - y_1x_2)$$

### 4.3 Functions of a Complex Variable in Instantiation Space

Complex functions of a complex variable are designated

$$w = f(z) = f(x + iy), \quad x, y \in R$$

It is customary to write complex functions in terms of their real and imaginary parts i.e.

$$w = f(z) = u + iv, \quad u = u(x, y) \in R, \quad v = v(x, y) \in R$$

Geometrically, a complex function sends points in the $z$-plane into points in the $w$-plane. A complex function can be represented in instantiation space by a projection i.e.

$$f(z) = u + iv, \quad u = u(x, y), \quad v = v(x, y),$$

$$v = v(x, y) \rightarrow \overline{f(\bar{x} + iv)} = \bar{u} + i\bar{v} \rightarrow u + iv \rightarrow z \rightarrow \bar{z} \rightarrow \bar{u} + i\bar{v}$$

$$\begin{cases} (\bar{u}, \bar{v}), & \text{if } u \geq 0, \quad v \geq 0 \\ (\bar{u}, \bar{v}), & \text{if } u \leq 0, \quad v \leq 0 \\ (-|\bar{u}|, |\bar{v}|), & \text{if } u < 0, \quad v \geq 0' \\ (|\bar{u}|, -|\bar{v}|), & \text{if } u \geq 0, \quad v < 0 \end{cases}$$

where

$$\bar{u} = \sqrt{u^2 + v^2}, \quad \bar{v} = \sqrt{v^2 + u^2}, \quad |\bar{u}| = \sqrt{|u| + |v|}, \quad |\bar{v}| = \sqrt{|v| + |u|}$$

Example: Suppose $S \rightarrow C$, where the domain of definition is

$$\forall z(z = x + iy \in S \rightarrow 1 < x < 2), \quad f(z) = z^2 \in C,$$
then
\[ f(z) = (x + iy)^2 = x^2 - y^2 + i2xy, \quad u = x^2 - y^2, \quad v = 2xy \]
Since \( y = v/2x \), substituting \( v/2x \) for \( y \) in \( u \) leaves
\[ u = x^2 - \left( \frac{v}{2x} \right)^2 = x^2 - \frac{v^2}{4x^2} \]
The boundaries of the region in the domain of definition of \( f \) are the lines ‘\( x = 1 \)’ and ‘\( x = 2 \)’. On these boundaries, \( u = 1 - \frac{v^2}{4} \) and \( u = 4 - \frac{v^2}{16} \) respectfully. Hence, the line ‘\( x = 1 \)’ is mapped into the parabola ‘\( u = 1 - \frac{v^2}{4} \)’ by \( f \). And the line ‘\( x = 2 \)’ is mapped into the parabola ‘\( u = 4 - \frac{v^2}{16} \)’ (see figure 4.3-1).

\[
\begin{align*}
\bar{u} &= \sqrt{x^2 - \frac{v^2}{4x^2}} \sqrt{x^2 - \frac{v^2}{4x^2}} + v, \\
\bar{v} &= \sqrt{v} \sqrt{x^2 - \frac{v^2}{4x^2}} + v, \\
|\bar{u}| &= \sqrt{x^2 - \frac{v^2}{4x^2}} \sqrt{x^2 - \frac{v^2}{4x^2}} + |v|, \\
|\bar{v}| &= \sqrt{|v|} \sqrt{x^2 - \frac{v^2}{4x^2}} + |v|
\end{align*}
\]
Therefore, the horizontal strip in the \( z \)-plane is transformed into the parabolic strip in the \( w \)-plane \([190]\).

If \( f \) is projected into instantiation space, then
\[ f(z) = (x + iy)^2 = x^2 - y^2 + i2xy, \quad u = x^2 - \frac{v^2}{4x^2}, \]
\[ v = 2xy \rightarrow \bar{f}(\bar{z}) = \begin{cases} 
\{\bar{u}, \bar{v}\}, & \text{if } u \geq 0, \quad v \geq 0 \\
\{\bar{u}, \bar{v}\}, & \text{if } u \leq 0, \quad v \leq 0 \\
\{-|\bar{u}|, |\bar{v}|\}, & \text{if } u < 0, \quad v \geq 0' \\
\{|\bar{u}|, -|\bar{v}|\}, & \text{if } u \geq 0, \quad v < 0 
\end{cases} \]
4.3.1 Multivalued Functions in Instantiation Space

The complex function ‘\( f(z) \)’ is multivalued when points in the domain of \( f \) are transformed into more than one point in the \( w \)-plane. Moreover, it is convenient to consider an idealized point called a ‘point at infinity’. For instance, if \( w = 1/z \), then the point ‘\( z = 0 \)’ is transformed into the point ‘\( w = \infty \)’, allowing some functions, discontinuous at a point, to become continuous at that point.

The following limits can be defined:

\[
\lim_{z \to \infty} f(z) = l \equiv \forall z \exists \epsilon \exists M (\epsilon, M > 0 \land |z| > M \Rightarrow |f(z) - l| < \epsilon)
\]

\[
\lim_{z \to z_0} f(z) = \infty \equiv \exists \delta \forall N (\delta, N > 0 \land 0 < |z - z_0| < \delta \Rightarrow |f(z)| > N)
\]

4.3.1.1 Branch Points

Since \( \arg z \) is not unique,

\[ z = r[\cos(\theta + 2\pi k) + i \sin(\theta + 2\pi k)], \quad k = 0, 1, 2, ..., n - 1 \]

represents the same point for different values of \( k \). However,

\[ z = r(\cos \theta + i \sin \theta) = r[\cos(\theta + 2\pi) + i \sin(\theta + 2\pi)] \rightarrow f[r(\cos(\theta) + i \sin(\theta))] \neq f[r(\cos(\theta + 2\pi) + i \sin(\theta + 2\pi))] \]

In this case, \( f \) is ‘multivalued’. A branch point of a multivalued function is a point where the function becomes discontinuous when \( z \) is allowed to encircle an arbitrary point.

Example: Let \( f(z) = \ln z \). If

\[ z = e^w = r(\cos \theta + i \sin \theta) = e^{u+iv} = e^u e^{iv} = e^u (\cos v + i \sin v) \rightarrow e^u \cos v = r \cos \theta, \]

\[ e^u \sin v = r \sin \theta, \quad w = u + iv \]

Hence,

\[ e^u = r \rightarrow \ln e^u = \ln r \rightarrow u = \ln r, \quad r \cos v = r \cos \theta, \]

\[ r \sin v = r \sin \theta \rightarrow v = \theta \pm 2\pi k \rightarrow w = u + iv = \ln r + i(\theta \pm 2\pi k) \]

If \( z = e^w \), then \( w = \ln z \). Let

\[ \ln z_1 = \ln r_1 + i \theta_1 \neq 0, \]

then making one revolution around about the point ‘\( z = 0 \)’, and returning to the point ‘\( z_1 \neq 0 \)’, implies that

\[ z_1 = r_1 e^{i\theta_1} = r_1 e^{i(\theta_1 + 2\pi)} \]

But

\[ \ln r_1 + i \theta_1 \neq \ln r_1 + i(\theta_1 + 2\pi) \]
After one revolution, \( f \) is on a different ‘branch’. The point ‘\( z = 0 \)’ is called a ‘branch point’. If a function is restricted to one of its branches, it remains single valued.

Hence, if \( z(r, \theta) \) is a branch point, then

\[
\begin{align*}
    z = re^{i\theta} \rightarrow f(z) = \ln z = \ln(re^{i\theta}) = \ln r + \ln e^{i\theta} = \ln r + i\theta,
\end{align*}
\]

In this case, \( u = \ln r \) and \( v = \theta \). Therefore,

\[
\ln z = \begin{cases}
    \langle \tilde{u}, \tilde{v} \rangle, & \text{if } u \geq 0, \quad v \geq 0 \\
    \langle \tilde{u}, \tilde{v} \rangle, & \text{if } u \leq 0, \quad v \leq 0 \\
    \langle -|\tilde{u}|, |\tilde{v}| \rangle, & \text{if } u < 0, \quad v \geq 0' \\
    \langle |\tilde{u}|, -|\tilde{v}| \rangle, & \text{if } u \geq 0, \quad v < 0
\end{cases}
\]

where

\[
\tilde{u} = \sqrt{\ln r} \sqrt{\ln r + \theta}, \quad \tilde{v} = \sqrt{\theta} \sqrt{\ln r + \theta}, \quad |\tilde{u}| = \sqrt{|\ln r| \sqrt{\ln r + |\theta|}}, \quad r > 0,
\]

and \( \ln z \) is \( \ln z \) instantiated. Note that \( \ln z \neq \ln \tilde{z} \).

### 4.4 Complex Differentiation in Instantiation Space

A complex function ‘\( f(z) \)’ is ‘analytic’ at ‘\( z_0 \)’ if there exists a neighborhood

\[
|z - z_0| < \delta, \quad \delta > 0,
\]

all the points of which \( f'(z) = df/dz \) exists. If \( f'(z) \) exists at all points in a region ‘\( \mathcal{R} \)’, then \( f \) is analytic in \( \mathcal{R} \).

#### 4.4.1 The Cauchy-Riemann Equations Instantiated

A necessary condition for

\[
f(z) = u(x, y) + iv(x, y)
\]

to be analytic in a region ‘\( \mathcal{R} \)’ is

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}
\]

These are the ‘Cauchy-Riemann’ equations. If the partial derivatives are continuous in \( \mathcal{R} \), then the Cauchy-Riemann equations are also sufficient conditions. Moreover, if both \( u \) an \( v \) have continuous second order derivatives, then

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} \right) = \frac{\partial}{\partial x} \left( \frac{\partial v}{\partial y} \right) + \frac{\partial}{\partial y} \left( -\frac{\partial v}{\partial x} \right) = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0,
\]
\[
\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial x} \left( \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{\partial v}{\partial y} \right) = -\frac{\partial}{\partial x} \left( \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} \right) = -\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial x} = 0
\]

Let
\[
\frac{\tilde{\partial}}{\tilde{\partial} x} = \sqrt{\frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \left( \frac{\partial}{\partial x} \right)^2 + \left( \frac{\partial}{\partial y} \right)^2},
\]

\[
\left( \frac{\tilde{\partial}}{\tilde{\partial} x} \right)^2 \tilde{u} = \left[ \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial y} \frac{\partial}{\partial x} \right] \tilde{u} = \left[ \frac{\partial^2 \tilde{u}}{\partial x^2} + \frac{\partial}{\partial y} \left( \frac{\partial \tilde{u}}{\partial x} \right) \right] = \left[ \frac{\partial^2 \tilde{u}}{\partial x^2} + \frac{\partial}{\partial y} \left( \frac{\partial \tilde{u}}{\partial x} \right) \right] = \left[ \frac{\partial^2 \tilde{u}}{\partial x^2} + \frac{\partial^2 \tilde{v}}{\partial y^2} \right]
\]

Similarly,
\[
\left( \frac{\tilde{\partial}}{\tilde{\partial} y} \right)^2 \tilde{v} = \left[ \frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial x} \frac{\partial}{\partial y} \right] \tilde{v} = \left[ \frac{\partial^2 \tilde{v}}{\partial y^2} + \frac{\partial}{\partial x} \left( \frac{\partial \tilde{v}}{\partial y} \right) \right] = \left[ \frac{\partial^2 \tilde{v}}{\partial y^2} + \frac{\partial}{\partial x} \left( \frac{\partial \tilde{v}}{\partial y} \right) \right] = \left[ \frac{\partial^2 \tilde{v}}{\partial y^2} + \frac{\partial^2 \tilde{u}}{\partial x^2} \right]
\]

Hence,
\[
\left( \frac{\tilde{\partial}}{\tilde{\partial} x} \right)^2 \tilde{u} = \left( \frac{\tilde{\partial}}{\tilde{\partial} y} \right)^2 \tilde{v}, \quad \left( \frac{\tilde{\partial}}{\tilde{\partial} y} \right)^2 \tilde{u} = -\left( \frac{\tilde{\partial}}{\tilde{\partial} x} \right)^2 \tilde{v}
\]

These are the Cauchy-Riemann equations instantiated. The analysis above leads to the following conclusion:

*If* \( f(z) = u + iv \) *is analytic in a region* \( \mathcal{R} \in \mathcal{C} \), \( \tilde{f}(z) = \tilde{u} + i \tilde{v} \) *and*

\[
\left( \frac{\tilde{\partial}}{\tilde{\partial} x} \right)^2 \tilde{u} = \left( \frac{\tilde{\partial}}{\tilde{\partial} y} \right)^2 \tilde{v}, \quad \left( \frac{\tilde{\partial}}{\tilde{\partial} y} \right)^2 \tilde{u} = -\left( \frac{\tilde{\partial}}{\tilde{\partial} x} \right)^2 \tilde{v},
\]

*then* \( \tilde{f}(z) \) *is analytic in* \( \tilde{\mathcal{R}} \in \tilde{\mathcal{C}} \).

### 4.4.2 The Jacobian

The functions \('u = u(x, y)' and 'v = v(x, y)' represent transformations or mappings that
establish a correspondence between points in the \( u, v \)- and \( x, y \)-planes respectively. If such mappings are one-to-one and onto, then for each point in the \( u, v \)-plane, there corresponds one and only one point in the \( x, y \)-plane. Under these conditions, a closed
region 'ℛ' in the $x, y$-plane, in general, is mapped to a closed region 'ℛ'' in the $u, v$-plane. If $\Delta A_{xy}$ and $\Delta A_{uv}$ represent the areas of these regions and if $u$ and $v$ are continuously differentiable in those regions, then

$$\lim_{\Delta A_{uv}, \Delta A_{xy} \to 0} \frac{\Delta A_{uv}}{\Delta A_{xy}} = \left| \frac{\partial (u, v)}{\partial (x, y)} \right|$$

The factor '|$\partial (u, v)/\partial (x, y)|'$ is called the 'Jacobian' of the transformation (see Book II: Sec. 10.6.3.3.1) i.e.

$$\frac{\partial (u, v)}{\partial (x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x}$$

If solving $u = u(x, y)$ and $v = v(x, y)$ for $x$ and $y$ leaves $x = x(u, v)$ and $y = y(u, v)$, where $x$ and $y$ are single-valued and continuously differentiable, then

$$\frac{\partial (u, v)}{\partial (x, y)} = \frac{\partial (x, y)}{\partial (u, v)}$$

To see this, note that

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy, \quad dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv, \quad dy = \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv$$

Hence,

$$du = \frac{\partial u}{\partial x} \left( \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv \right) + \frac{\partial u}{\partial y} \left( \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv \right)$$

$$= \left( \frac{\partial u}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial u} \right) du + \left( \frac{\partial u}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial v} \right) dv$$

Therefore,

$$\frac{\partial u}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial u} = 1, \quad \frac{\partial u}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial v} = 0$$

Similarly,

$$\frac{\partial v}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial u} = 0, \quad \frac{\partial v}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial v} = 1$$
So,

\[
\frac{\partial (u, v)}{\partial (x, y)} \cdot \frac{\partial (x, y)}{\partial (u, v)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial u} - \frac{\partial v}{\partial x} \frac{\partial x}{\partial v} - \frac{\partial v}{\partial y} \frac{\partial y}{\partial v} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = I
\]

If \( u \) and \( v \) are the ‘real’ and ‘imaginary’ parts of an analytic function ‘\( f(z) \)’ respectively, then

\[
\frac{\partial (u, v)}{\partial (x, y)} = |f'(z)|^2
\]

To see this, if \( f(z) \) is analytic in \( R \), then the Cauchy-Riemann equations

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}
\]

are satisfied. Hence,

\[
\frac{\partial (u, v)}{\partial (x, y)} = \left| \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} \right| = |f'(z)|^2
\]

The transformation is one-to-one if \( f'(z) \neq 0 \) for all points in the region \([191]\).

**4.4.2.1 The Derivative of a Function in Instantiation Space**

Suppose \( f \) is analytic at \( z \) and

\[
f(z) = u(x, y) + iv(x, y)
\]

The derivative of \( f \) at \( z \) can be computed using the definition of a limit either along the real or the imaginary axis. Along the real axis,

\[
f'(z) = \lim_{h \to 0} \frac{f(z + h) - f(z)}{h} = \lim_{h \to 0} \frac{f(x + iy + h) - f(x + iy)}{h} = \lim_{h \to 0} \frac{u(x + h, y) + iv(x + h, y) - u(x, y) - iv(x, y)}{h}
\]

Separating the last expression into its real and imaginary parts,

\[
f'(z) = \lim_{h \to 0} \left( \frac{u(x + h, y) - u(x, y)}{h} + i \frac{v(x + h, y) - v(x, y)}{h} \right)
\]

The two terms on the right-side of the equation above are \( \partial u / \partial x \) and \( \partial v / \partial x \) respectfully. So, if \( f \) is analytic at \( z \), then \( \partial u / \partial x \) and \( \partial v / \partial x \) exist and
\[
f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}
\]

If \( h \to 0 \) along the imaginary axis, then

\[
f'(z) = \frac{\partial u}{\partial y} - i \frac{\partial v}{\partial y}
\]

If \( f \) is analytic at \( z \), then

\[
f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \to f'(z) = \begin{cases} 
\langle \bar{u}, \bar{v} \rangle, & \text{if } \frac{\partial u}{\partial x} \geq 0, \quad \frac{\partial v}{\partial x} \geq 0 \\
\langle \bar{u}, \bar{v} \rangle, & \text{if } \frac{\partial u}{\partial x} \leq 0, \quad \frac{\partial v}{\partial x} \leq 0 \\
\langle -|\bar{u}|, |\bar{v}| \rangle, & \text{if } \frac{\partial u}{\partial x} < 0, \quad \frac{\partial v}{\partial x} \geq 0 \\
\langle |\bar{u}|, -|\bar{v}| \rangle, & \text{if } \frac{\partial u}{\partial x} \geq 0, \quad \frac{\partial v}{\partial x} < 0 
\end{cases}
\]

where

\[
\bar{u} = \sqrt{\frac{\partial u}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \frac{\partial v}{\partial x}}, \quad \bar{v} = \sqrt{\frac{\partial v}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x} \frac{\partial u}{\partial x}}, \quad |\bar{u}| = \sqrt{\frac{\partial u}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \frac{\partial v}{\partial x}},
\]

\[
|\bar{v}| = \sqrt{\frac{\partial v}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x} \frac{\partial u}{\partial x}}.
\]

Alternatively,

\[
f'(z) = \frac{\partial u}{\partial y} - i \left( \frac{\partial v}{\partial y} \right) \to f'(z) = \begin{cases} 
\langle \bar{u}, \bar{v} \rangle, & \text{if } \frac{\partial u}{\partial y} \geq 0, \quad -\frac{\partial v}{\partial y} \geq 0 \\
\langle \bar{u}, \bar{v} \rangle, & \text{if } \frac{\partial u}{\partial y} \leq 0, \quad -\frac{\partial v}{\partial y} \leq 0 \\
\langle -|\bar{u}|, |\bar{v}| \rangle, & \text{if } \frac{\partial u}{\partial y} < 0, \quad -\frac{\partial v}{\partial y} \geq 0 \\
\langle |\bar{u}|, -|\bar{v}| \rangle, & \text{if } \frac{\partial u}{\partial y} \geq 0, \quad -\frac{\partial v}{\partial y} < 0 
\end{cases}
\]

where

\[
\bar{u} = \sqrt{\frac{\partial u}{\partial y} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \frac{\partial v}{\partial y}}, \quad \bar{v} = \sqrt{\frac{\partial v}{\partial y} \frac{\partial v}{\partial y} + \frac{\partial u}{\partial y} \frac{\partial u}{\partial y}}, \quad |\bar{u}| = \sqrt{\frac{\partial u}{\partial y} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \frac{\partial v}{\partial y}},
\]

\[
|\bar{v}| = \sqrt{\frac{\partial v}{\partial y} \frac{\partial v}{\partial y} + \frac{\partial u}{\partial y} \frac{\partial u}{\partial y}}.
\]
where \( f'(z) \) is \( f'(z) \) instantiated.

### 4.4.2.2 The Gradient Instantiated

Recall that if \( f \) is any continuously differentiable function of a complex variable, then

\[
\nabla f = \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y}
\]

is called the 'gradient' of \( f \), which is a vector normal to the curve

\[
f(x, y) = c, \quad c = a \text{ constant}
\]

Hence,

\[
\nabla f = \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \rightarrow \nabla f = \begin{cases} 
\langle \bar{u}, \bar{v} \rangle, & \text{if } \frac{\partial f}{\partial x} \geq 0, \quad \frac{\partial f}{\partial y} \geq 0 \\
\langle \bar{u}, \bar{v} \rangle, & \text{if } \frac{\partial f}{\partial x} \leq 0, \quad \frac{\partial f}{\partial y} \leq 0 \\
\langle -|\bar{u}|, |\bar{v}| \rangle, & \text{if } \frac{\partial f}{\partial x} < 0, \quad \frac{\partial f}{\partial y} \geq 0 \\
\langle |\bar{u}|, -|\bar{v}| \rangle, & \text{if } \frac{\partial f}{\partial x} \geq 0, \quad \frac{\partial f}{\partial y} < 0
\end{cases}
\]

where

\[
\bar{u} = \sqrt{\frac{\partial f}{\partial x} \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial f}{\partial y}}, \quad \bar{v} = \sqrt{\frac{\partial f}{\partial y} \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial f}{\partial y}}, \quad |\bar{u}| = \sqrt{\frac{\partial f}{\partial x} \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial f}{\partial y}}
\]

where \( \nabla f \) is the 'gradient' of \( f \) instantiated. Similarly, the gradient of a complex function '\( a = P + iQ, \ P, Q \in \mathbb{R} \)' is

\[
\nabla a = \frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} + i \left( \frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} \right) \rightarrow \nabla a = \begin{cases} 
\langle \bar{u}, \bar{v} \rangle, & \text{if } \frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} \geq 0, \quad \frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} \geq 0 \\
\langle \bar{u}, \bar{v} \rangle, & \text{if } \frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} \leq 0, \quad \frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} \leq 0 \\
\langle -|\bar{u}|, |\bar{v}| \rangle, & \text{if } \frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} < 0, \quad \frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} \geq 0 \\
\langle |\bar{u}|, -|\bar{v}| \rangle, & \text{if } \frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} \geq 0, \quad \frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} < 0
\end{cases}
\]
where
\[
\tilde{u} = \sqrt{\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}} \sqrt{\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} + \frac{\partial Q}{\partial x}} \quad \tilde{v} = \sqrt{\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}} \sqrt{\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} + \frac{\partial Q}{\partial x}},
\]
\[
|\tilde{u}| = \sqrt{\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}} \sqrt{\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} + \frac{\partial Q}{\partial x}},
\]
\[
|\tilde{v}| = \sqrt{\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}} \sqrt{\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} + \frac{\partial Q}{\partial x}}.
\]

4.4.2.3 The Divergence Instantiated

Now
\[
\langle \nabla | a \rangle = Re \left[ \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) (P + iQ) \right] = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \rightarrow |\nabla|a| = \sqrt{\left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right), 0} \sqrt{\left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) + 0} = \langle \nabla | a \rangle,
\]
where \(\langle \nabla | a \rangle\) is the ‘divergence’ of \(a\) instantiated. Note that \(\langle \nabla | a \rangle\) is a scalar and \(\langle \nabla | a \rangle = \overline{\langle \nabla | a \rangle}\).

4.4.2.4 The Curl Instantiated

Next, let
\[
\langle \nabla \times | a \rangle = \mathcal{I}_m \left[ \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) (P + iQ) \right] = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \rightarrow |\nabla \times | a \rangle
\]
\[
= \left( 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \sqrt{0 + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) = \partial Q - \partial P}.
\]
then \(\langle \nabla \times | a \rangle\) is called the ‘curl’ of \(a\) instantiated. Note that
\[
\overline{\langle \nabla \times | a \rangle} = \langle \nabla \times | a \rangle
\]
4.4.2.5 The Laplacian Instantiated

Finally, let
\[
\langle \nabla | \nabla \rangle = \nabla^2 = Re \left[ \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \right] = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \rightarrow \nabla^2
\]

\[
= \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right), 0 \right) \sqrt{\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + 0} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},
\]

then \( \nabla^2 \) is called the ‘Laplacian’ instantiated. If \( a \) is analytic, \( \nabla^2 a = 0 \) i.e. \( \nabla^2 P = 0 \) and \( \nabla^2 Q = 0 \), then \( P \) and \( Q \) are called ‘harmonic’. Note that \( \nabla^2 = \nabla^2 \).

4.5 Complex Curves in Instantiation Space

Integration of functions of a complex variable is carried out over curves in \( C \). If \( \alpha(t) \) and \( \beta(t) \) are functions of a real variable, \( t \in R \), a single parameter, and \( \alpha(t) \) and \( \beta(t) \) are continuous in \( t_1 \leq t \leq t_2 \), then
\[
z(t) = \alpha(t) + i\beta(t)
\]
defines a continuous curve or arc in the plane joining the points ‘\( z(t_1) \)’ and ‘\( z(t_2) \)’. In terms of differential geometry, \( z(t) \) is a ‘rectifiable’ curve, a curve with a definite length.

The function ‘\( z(t) \)’ is continuous at ‘\( t_0 \)’ if both \( \alpha(t) \) and \( \beta(t) \) are continuous there and is continuous in \( t_1 \leq t \leq t_2 \) if both \( \alpha(t) \) and \( \beta(t) \) are. A function ‘\( z(t) \)’ is differentiable at ‘\( t_0 \)’ if
\[
z'(t_0) = \alpha'(t_0) + i\beta'(t_0)
\]
exists.

‘\( z(t) \)’ can be considered a position vector whose terminal point draws out a curve ‘\( C \)’ as \( t \) varies from \( t_1 \) to \( t_2 \). If \( z(t) \) and \( z(t + \Delta t) \) represent two points on the curve, then
\[
\frac{z(t + \Delta t) - z(t)}{\Delta t} = \frac{\Delta z}{\Delta t}, \quad \Delta z = z(t + \Delta t) - z(t)
\]
If
\[
\lim_{\Delta t \to 0} \frac{z(t + \Delta t) - z(t)}{\Delta t} = \frac{dz}{dt}
\]
exists, then \( dz/dt \) is a vector in the direction of the tangent to the curve and is given by
\[
\frac{dz}{dt} = \frac{dx}{dt} + i\frac{dy}{dt}
\]
Hence,

\[ z'(t) = \frac{dz}{dt} = \frac{d\alpha}{dt} + i \frac{d\beta}{dt} \rightarrow \frac{\overline{dz}}{dt} = \begin{cases} 
\langle \overline{u}, \overline{v} \rangle, & \text{if } \frac{d\alpha}{dt} \geq 0, \quad \frac{d\beta}{dt} \geq 0 \\
\langle \overline{u}, \overline{v} \rangle, & \text{if } \frac{d\alpha}{dt} \leq 0, \quad \frac{d\beta}{dt} \leq 0 \\
\langle -|\overline{u}|, |\overline{v}| \rangle, & \text{if } \frac{d\alpha}{dt} < 0, \quad \frac{d\beta}{dt} \geq 0 \\
\langle |\overline{u}|, -|\overline{v}| \rangle, & \text{if } \frac{d\alpha}{dt} \geq 0, \quad \frac{d\beta}{dt} < 0 
\end{cases} \]

where

\[
\overline{u} = \sqrt{\frac{d\alpha}{dt} + \frac{d\beta}{dt}}, \quad \overline{v} = \sqrt{\frac{d\alpha}{dt} + \frac{d\beta}{dt}}, \quad |\overline{u}| = \sqrt{\frac{|d\alpha|}{dt} + \frac{|d\beta|}{dt}}, \quad |\overline{v}| = \sqrt{\frac{|d\alpha|}{dt} + \frac{|d\beta|}{dt}},
\]

given that \( \alpha(t) \) and \( \beta(t) \) are differentiable. So, \( \overline{dz}/dt \) is \( dz/dt \) instantiated.

### 4.6 Conformal Mappings

A curve is called ‘simple’ if \( z(t_i) \neq z(t_j), \ i \neq j \) for all \( i \) and \( j \) whenever \( t_i \neq t_j, \ t_1 < t_i, \ t_j < t_2 \). If \( z(t) \) is simple in \( t_1 \leq t \leq t_2 \) and \( z(t_1) = z(t_2) \), the curve is ‘closed’ (see fig. 4.6-1) [190].

![Figure 4.6-1](image)

If, under the transformation \( u = u(x, y), \ v = v(x, y) \), the point \( (x_0, y_0) \) in the \( x, y \)-plane is mapped to the point \( (u_0, v_0) \) while the points along the curves \( C_1 \) and \( C_2 \) that intersect at \( (x_0, y_0) \) are mapped into the curves \( C_1' \) and \( C_2' \) in the \( u, v \)-plane and if the angle between \( C_1 \) and \( C_2 \) at \( (x_0, y_0) \) is equal both in magnitude and sense to the angle
between \( C'_1 \) and \( C'_2 \) at \((u_0, v_0)\), then the transformation is 'conformal'. The following theorem is fundamental:

*If \( f(z) \) is analytic in a region ‘\( \mathcal{R} \)’, then \( f(z) \) is conformal.*

Suppose \( C \) is a simple closed curve in the \( x, y \)-plane forming a region ‘\( \mathcal{R} \)’. Further suppose \( C' \) is a simple closed curve forming a circle of radius ‘1’ with center at the origin of the \( u, v \)-plane encircling the region ‘\( \mathcal{R}' \)’. The following theorem is fundamental [191]:

*There exists a one-to-one and onto analytic function ‘\( f(z) \)’ which maps each point in \( \mathcal{R} \) into a point in \( \mathcal{R}' \) and each point on \( C \) to a corresponding point on \( C' \).*

### 4.6.1 Complex Integration in Instantiation Space

Suppose \( C \) is a rectifiable arc described by the function

\[
    z(t) = \alpha(t) + i\beta(t)
\]

and let the complex valued function ‘\( f(z) = f(z(t)) \)’ be continuous on \( C \) in \( a \leq t \leq b \). The 'line integral' of \( f \) on \( C \) is defined as

\[
    \int_C f(z) \, dz = \int_a^b f(z(t)) \, z'(t) \, dt, \quad dz = z'(t) \, dt
\]

The integral of \( f(z) \) is sensitive to the direction along which the curve is traversed:

\[
    \int_a^b f(z(t)) \, z'(t) \, dt = -\int_b^a f(z(t)) \, z'(t) \, dt
\]

If a function ‘\( F \)’ is analytic in a region ‘\( \mathcal{R} \)’, has continuous derivatives, \( dF = f \, dz \) in \( \mathcal{R} \) and if an arc ‘\( C \)’ with endpoints ‘\( z_1 \)’ and ‘\( z_2 \)’ is totally contained in \( \mathcal{R} \), then

\[
    \int_C f(z) \, dz = \int_{z_1}^{z_2} f(z) \, dz = F(z_2) - F(z_1),
\]

by the fundamental theorem of the calculus. Note that if \( F \) is analytic in \( \mathcal{R} \), then

\[
    \int_C f(z) \, dz
\]

is independent of the contour ‘\( C \)’ which lies in \( \mathcal{R} \). Moreover, if \( z_1 = z_2 \), then

\[
    \int_C f(z) \, dz = F(z_2) - F(z_1) = 0
\]

The function ‘\( z(t) \)’ is integrable over \( t_1 \leq t \leq t_2 \) if both \( \alpha(t) \) and \( \beta(t) \) are.
Hence,
\[ \int_{t_1}^{t_2} z(t) \, dt = \int_{t_1}^{t_2} \alpha(t) \, dt + i \int_{t_1}^{t_2} \beta(t) \, dt \to \int_{t_1}^{t_2} z(t) \, dt \]

\[ = \begin{cases} 
\langle \bar{u}, \bar{v} \rangle, & \text{if } \int_{t_1}^{t_2} \alpha(t) \, dt \geq 0, \quad \int_{t_1}^{t_2} \beta(t) \, dt \geq 0 \\
\langle \bar{u}, \bar{v} \rangle, & \text{if } \int_{t_1}^{t_2} \alpha(t) \, dt \leq 0, \quad \int_{t_1}^{t_2} \beta(t) \, dt \leq 0 \\
\langle -|\bar{u}|, |\bar{v}| \rangle, & \text{if } \int_{t_1}^{t_2} \alpha(t) \, dt < 0, \quad \int_{t_1}^{t_2} \beta(t) \, dt \geq 0 \\
\langle |\bar{u}|, -|\bar{v}| \rangle, & \text{if } \int_{t_1}^{t_2} \alpha(t) \, dt \geq 0, \quad \int_{t_1}^{t_2} \beta(t) \, dt < 0 
\end{cases} \]

where
\[ \bar{u} = \sqrt{\int_{t_1}^{t_2} \alpha(t) \, dt} \sqrt{\int_{t_1}^{t_2} \alpha(t) \, dt + \int_{t_1}^{t_2} \beta(t) \, dt}, \]
\[ \bar{v} = \sqrt{\int_{t_1}^{t_2} \beta(t) \, dt} \sqrt{\int_{t_1}^{t_2} \alpha(t) \, dt + \int_{t_1}^{t_2} \beta(t) \, dt}, \]
\[ |\bar{u}| = \sqrt{\int_{t_1}^{t_2} \alpha(t) \, dt} \sqrt{\int_{t_1}^{t_2} \alpha(t) \, dt + \int_{t_1}^{t_2} \beta(t) \, dt}, \]
\[ |\bar{v}| = \sqrt{\int_{t_1}^{t_2} \beta(t) \, dt} \sqrt{\int_{t_1}^{t_2} \alpha(t) \, dt + \int_{t_1}^{t_2} \beta(t) \, dt}, \]

where \( \int_{t_1}^{t_2} z(t) \, dt \) is \( \int_{t_1}^{t_2} z(t) \, dt \) instantiated.

4.6.1.1 The Connection between Real and Complex Line Integrals

There is a connection between real and complex line integrals. If
\[ f(z) = u(x, y) + iv(x, y), \quad x, y \in \mathbb{R}, \]

then \( u \) and \( v \) are functions of the real variables ‘\( x \)’ and ‘\( y \)’. Hence,
\[ \int_{c} f(z) \, dz = \int_{c} (u + iv) \, (dx + idy) = \int_{c} u \, dx - v \, dy + i \int_{c} v \, dx + u \, dy \to \int_{c} f(z) \, dz \]
\[
\langle \bar{u}, \bar{v} \rangle, \quad \text{if} \quad \int_C u \, dx - v \, dy \geq 0, \quad \int_C v \, dx + u \, dy \geq 0
\]
\[
\langle \bar{u}, \bar{v} \rangle, \quad \text{if} \quad \int_C u \, dx - v \, dy \leq 0, \quad \int_C v \, dx + u \, dy \leq 0
\]
\[
\langle -|\bar{u}|, |\bar{v}| \rangle, \quad \text{if} \quad \int_C u \, dx - v \, dy < 0, \quad \int_C v \, dx + u \, dy \geq 0
\]
\[
\langle |\bar{u}|, -|\bar{v}| \rangle, \quad \text{if} \quad \int_C u \, dx - v \, dy \geq 0, \quad \int_C v \, dx + u \, dy < 0
\]

where
\[
\bar{u} = \sqrt{\int_C u \, dx - v \, dy} \sqrt{\int_C u \, dx - v \, dy + \int_C v \, dx + u \, dy},
\]
\[
\bar{v} = \sqrt{\int_C v \, dx + u \, dy} \sqrt{\int_C u \, dx - v \, dy + \int_C v \, dx + u \, dy},
\]
\[
|\bar{u}| = \sqrt{\int_C u \, dx - v \, dy} \sqrt{\int_C u \, dx - v \, dy + \int_C v \, dx + u \, dy},
\]
\[
|\bar{v}| = \sqrt{\int_C v \, dx + u \, dy} \sqrt{\int_C u \, dx - v \, dy + \int_C v \, dx + u \, dy},
\]

where \( \int_C f(z) \, dz \) is \( \int_C f(z) \, dz \) instantiated.

**4.7 The Length of a Line in Instantiation Space**

Let \( z(t) = x(t) + iy(t) \) be analytic, then define
\[
|z'(t)| = \frac{dx}{dt} + i \frac{dy}{dt} = \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} = \frac{ds}{dt} \to |z'(t)|
\]
\[
= \sqrt{\frac{dx}{dt} \left( \frac{dx}{dt} + \frac{dy}{dt} \right) + i \frac{dy}{dt} \left( \frac{dx}{dt} + \frac{dy}{dt} \right)} \equiv \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 + 2i \frac{dy}{dt} \frac{dx}{dt}}
\]
\[
= \frac{ds}{dt}
\]

where \( \frac{ds}{dt} \) is \( ds/dt \) instantiated. The function \( 's(t)' \) gives the length \( 'L' \) of the arc \( 'C' \) from \( z(a) \) to \( z(t) \). Hence,
\[
\int_c |z'(t)| \, dt = \int_c \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt = L \rightarrow \int_c |z'(t)| \, dt
\]

where \(L\) is the length of the arc instantiated. Moreover,

\[
\frac{\overline{ds}}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + 2 \left| \frac{dx}{dt} \right| \left| \frac{dy}{dt} \right|} \rightarrow \overline{ds}^2 = dx^2 + dy^2 + 2|dx||dy|
\]

However, in some cases, the instantiation is invalid. This issue will be addressed in chapter 7.

4.8 Some Observations

Suppose there is a function

\[
f(x) = \left(\frac{1}{\sqrt[e]{e}}\right)^{1 - \sqrt{x}} \left(\sqrt[e]{e}\right)^{\sqrt{x}}
\]

If \(f(x)\) is expanded in a power series, then

\[
f(x) = e^{-\frac{1}{2}} + e^{-\frac{1}{2}} \sqrt{x} - \frac{1}{2} e^{-\frac{1}{2}} x + \frac{1}{6} e^{-\frac{1}{2}} x^3/2 + \frac{1}{24} e^{-\frac{1}{2}} x^2 + \frac{1}{120} e^{-\frac{1}{2}} x^{5/2} + \frac{1}{720} e^{-\frac{1}{2}} x^3 + \ldots
\]

\[
e^{-\frac{1}{2}} \left(1 + \sqrt{x} + \frac{1}{2} x + \frac{1}{6} x^{3/2} + \frac{1}{24} x^2 + \frac{1}{120} x^{5/2} + \frac{1}{720} x^3 + \frac{1}{5040} x^{7/2} + \ldots\right)
\]

Hence,

\[
\sqrt[e]{e} f(x) = \sqrt[e]{e} \left[\left(\frac{1}{\sqrt[e]{e}}\right)^{1 - \sqrt{x}} \left(\sqrt[e]{e}\right)^{\sqrt{x}}\right] = e^{\sqrt{x}}
\]

\[
= 1 + \sqrt{x} + \frac{1}{2!} x + \frac{1}{3!} x^{3/2} + \frac{1}{4!} x^2 + \frac{1}{5!} x^{5/2} + \frac{1}{6!} x^3 + \frac{1}{7!} x^{7/2} + \ldots
\]

Note that

\[
\cos(i\sqrt{x}) = 1 + \frac{1}{2!} x + \frac{1}{4!} x^2 + \frac{1}{6!} x^3 + \ldots,
\]

\[
-i \sin(i\sqrt{x}) = \sqrt{x} + \frac{1}{3!} x^{3/2} + \frac{1}{5!} x^{5/2} + \frac{1}{7!} x^{7/2} + \ldots
\]
Hence,

\[ \sqrt{e} f(x) = \sqrt{e} \left[ \left( \frac{1}{\sqrt{e}} \right)^{1-\sqrt{x}} \left( \sqrt{e} \right)^{\sqrt{x}} \right] = e^{\sqrt{x}} \approx \cos(i\sqrt{x}) - i \sin(i\sqrt{x}) \]

Now let

\[ a = \cos(i\sqrt{x}) = \cosh \sqrt{x} \rightarrow \cosh^{-1} a = \sqrt{x} \rightarrow \cosh^{-2} a = x \rightarrow \cosh^2(\cosh^{-2} a) = a \]

\[ = \cosh^2(x) \rightarrow \sqrt{a} = \pm \cosh x \]

Similarly, let

\[ b = -i \sin(i\sqrt{x}) = \sinh \sqrt{x} \rightarrow \sinh^{-1} b = \sqrt{x} \rightarrow \sinh^{-2} b = x \rightarrow \sinh^2(\sinh^{-2} b) = b \]

\[ = \sinh^2(x) \rightarrow \sqrt{b} = \pm \sinh x, \quad x \geq 0 \]

Therefore,

\[ -i\sqrt{b} = -i \sinh x = -\sin(i x), \quad \sqrt{a} = \cosh x = \cos(i x), \quad x \geq 0 \]

Hence, in instantiation space,

\[ \sqrt{e} f(x) = (\sqrt{a}, \sqrt{b}) \sqrt{a + b} = (\cos(i x), -\sin(i x)) \sqrt{\cos^2(i x) + \sin^2(i x)} = \cos(i x) - i \sin(i x), \]

since \( \cos^2(i x) + \sin^2(i x) = 1 \). Let \( i x = y \), then

\[ \cos i x - i \sin i x = \cos y - i \sin y = e^{-iy} = e^{-i(i x)} = e^x, \quad x \geq 0 \]

Moreover,

\[ \left[ \left( \frac{1}{\sqrt{e}} \right)^{1/2} \left( e^{1/2} \right)^x \right]^{1-x} = \left( \frac{1}{\sqrt{e}} \right)^{1-x} (\sqrt{e})^x \]

\[ = e^{-\frac{x}{2}} + e^{-\frac{x}{2}} x + \frac{1}{2} e^{-\frac{x}{2}} x^2 + \frac{1}{6} e^{-\frac{x}{2}} x^3 + \frac{1}{24} e^{-\frac{x}{2}} x^4 + \frac{1}{120} e^{-\frac{x}{2}} x^5 + \ldots \]

\[ = e^{-\frac{x}{2}} \left( 1 + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \frac{1}{5!} x^5 + \ldots \right) \rightarrow \sqrt{e} \left( \frac{1}{\sqrt{e}} \right)^{1-x} (\sqrt{e})^x \]

\[ = 1 + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \frac{1}{5!} x^5 = e^x \]

Let \( x \rightarrow ix \), then

\[ \sqrt{e} \left( \frac{1}{\sqrt{e}} \right)^{1-ix} (\sqrt{e})^{ix} = e^{ix} = \cos x + i \sin x, \]

which is Euler's formula.
Now,

\[
\left( \left[ \left( \frac{1}{e} \right)^{1/2} \right]^{1-x} \left( e^{1/2} \right)^x \right)^2 = e \left( \frac{1}{e} \right)^{1-x} e^x = e^{2x}
\]

In general, if \( e \) is replaced by any other real number \( 'G > 0' \), then, evidently,

\[
G \left( \frac{1}{G} \right)^{1-x} G^x = G^{x} \rightarrow \sqrt{G} \left( \frac{1}{\sqrt{G}} \right)^{1-x} (\sqrt{G})^x = G^x
\]

Now consider

\[
g(x) = \left( \frac{-1}{\sqrt{e}} \right)^{-1+x} \left( \sqrt{e} \right)^{-x}
\]

Expanding \( g(x) \) in a power series gives

\[
g(x) = \left( \frac{-1}{\sqrt{e}} \right)^{-1+x} \left( \sqrt{e} \right)^{-x} = -\sqrt{e} - \sqrt{e}(-1 + i\pi)x + \frac{\sqrt{e}(-1 + 2i + \pi^2)x^2}{2!} + \frac{\sqrt{e}(1 - 3i\pi - 3\pi^2 + i\pi^3)x^3}{3!} - \frac{\sqrt{e}(1 - 4i\pi - 6\pi^2 = 4i\pi^3 + \pi^4)x^4}{4!} - \frac{\sqrt{e}(-1 + 5i\pi + 10\pi^2 - 10i\pi^3 - 5\pi^4 + i\pi^5)x^5}{5!} + \ldots,
\]

which can be written

\[
g(x) = \left( \frac{-1}{\sqrt{e}} \right)^{-1+x} \left( \sqrt{e} \right)^{-x} = -\sqrt{e} - \sqrt{e}(-1 + i\pi)x - \frac{\sqrt{e}(-1 + i\pi)^2x^2}{2!} - \frac{\sqrt{e}(-1 + i\pi)^3x^3}{3!} - \frac{\sqrt{e}(-1 + i\pi)^4x^4}{4!} - \frac{\sqrt{e}(-1 + i\pi)^5x^5}{5!} + \ldots
\]

Let

\[
y = (-1 + i\pi)x,
\]
then
\[ g(x) = \left( \frac{-1}{\sqrt{e}} \right)^{-1+x} \left( \sqrt{e} \right)^{-x} = -\sqrt{e} \left( 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \cdots \right) = -\sqrt{e}e^y = -\sqrt{e}e^{(-1+in)x} \]

\[ \rightarrow \left( \frac{-1}{\sqrt{e}} \right)^{-1+x} \left( \sqrt{e} \right)^{-x} = -\sqrt{e}e^{-x}e^{inx} = -\sqrt{e} \left( \sqrt{e} \right)^{-x} e^{inx} \rightarrow \left( \frac{-1}{\sqrt{e}} \right)^{x} \]

\[ = \left( \sqrt{e} \right)^{-x} e^{inx} \rightarrow \left( \frac{-1}{\sqrt{e}} \right)^{x} \left( \sqrt{e} \right)^{x} = (-1)^x = e^{inx} = \cos(\pi x) + i \sin(\pi x) \]

Recall that
\[ \sqrt{e} \left( \frac{1}{\sqrt{e}} \right)^{1-ix} \left( \sqrt{e} \right)^{ix} = e^{ix} = \cos x + i \sin x \]

Upon replacing \( x \) with \( \pi x \), then
\[ \sqrt{e} \left( \frac{1}{\sqrt{e}} \right)^{1-inx} \left( \sqrt{e} \right)^{inx} = \cos(\pi x) + i \sin(\pi x) \rightarrow \sqrt{e} \left( \frac{1}{\sqrt{e}} \right)^{1-inx} \left( \sqrt{e} \right)^{inx} = \left( \frac{-1}{\sqrt{e}} \right)^{x} \left( \sqrt{e} \right)^{x} = (-1)^x \]

4.9 Concluding Remarks

If \( z = (x, y) \) is complex number, then \( \bar{z} \), its instantiation, can be specified by performing the following projection:

\[ z = (x, y) \rightarrow \bar{z} = \begin{cases} (\bar{x}, \bar{y}), & \text{if } x \geq 0, \quad y \geq 0 \\ (\bar{x}, \bar{y}), & \text{if } x \leq 0, \quad y \leq 0 \\ (-|\bar{x}|, |\bar{y}|), & \text{if } x < 0, \quad y \geq 0' \\ (|\bar{x}|, -|\bar{y}|), & \text{if } x \geq 0, \quad y < 0 \end{cases} \]

where
\[ \bar{x} = \sqrt{x} \sqrt{x + y}, \quad \bar{y} = \sqrt{y} \sqrt{x + y}, \quad |\bar{x}| = \sqrt{|x| \sqrt{|x| + |y|}}, \quad |\bar{y}| = \sqrt{|y| \sqrt{|x| + |y|}} \]

Complex functions of a single variable are designated
\[ w = f(z) = f(x + iy), \quad x, y \in R \rightarrow w = f(z) = u + iv, \quad u = u(x, y), \quad v = v(x, y) \]

Projection into instantiation is given by
\[ f(z) = u + iv, \quad u = u(x, y), \quad v = v(x, y) \rightarrow \]
\[ u + iv \xrightarrow{f} z \xrightarrow{\bar{}} \bar{u} + i\bar{v} = \begin{cases} (\bar{u}, \bar{v}), & \text{if } u \geq 0, \quad v \geq 0 \\ (\bar{u}, \bar{v}), & \text{if } u \leq 0, \quad v \leq 0 \\ (-|\bar{u}|, |\bar{v}|), & \text{if } u < 0, \quad v \geq 0' \\ (|\bar{u}|, -|\bar{v}|), & \text{if } u \geq 0, \quad v < 0 \end{cases} \]
where
\[ \bar{u} = \sqrt{u\bar{u}} + \bar{v}, \quad \bar{v} = \sqrt{v\bar{u}} + \bar{v}, \quad |\bar{u}| = \sqrt{|u||u| + |v|}, \quad |\bar{v}| = \sqrt{|v||u| + |v|} \]

Note that \( \bar{f} \) represents \( f \) instantiated.

If \( f \) is analytic at \( z \), then \( \frac{\partial u}{\partial x} \) and \( \frac{\partial v}{\partial x} \) exist at \( (x, y) \) and
\[
f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}
\]

Derivatives are instantiated by 'projection':
\[
f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \rightarrow f'(z) = \begin{cases} 
\langle \bar{u}, \bar{v} \rangle, & \text{if } \frac{\partial u}{\partial x} \geq 0, \quad \frac{\partial v}{\partial x} \geq 0 \\
\langle \bar{u}, \bar{v} \rangle, & \text{if } \frac{\partial u}{\partial x} \leq 0, \quad \frac{\partial v}{\partial x} \leq 0 \\
\langle -|\bar{u}|, |\bar{v}| \rangle, & \text{if } \frac{\partial u}{\partial x} < 0, \quad \frac{\partial v}{\partial x} \geq 0 \\
\langle |\bar{u}|, -|\bar{v}| \rangle, & \text{if } \frac{\partial u}{\partial x} \geq 0, \quad \frac{\partial v}{\partial x} < 0
\end{cases}
\]

where
\[ \bar{u} = \sqrt{\frac{\partial u}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \frac{\partial v}{\partial x}}, \quad \bar{v} = \sqrt{\frac{\partial v}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \frac{\partial v}{\partial x}}, \quad |\bar{u}| = \sqrt{|\frac{\partial u}{\partial x}| \sqrt{|\frac{\partial u}{\partial x}| + |\frac{\partial v}{\partial x}|}, \quad |\bar{v}| = \sqrt{|\frac{\partial v}{\partial x}| \sqrt{|\frac{\partial v}{\partial x}| + |\frac{\partial v}{\partial x}|} \]

The same procedure applies to integration. If
\[
z(t) = \alpha(t) + i\beta(t)
\]
is integrable over \( t_1 \leq t \leq t_2 \),
then

\[
\int_{t_1}^{t_2} z(t) \, dt = \int_{t_1}^{t_2} \alpha(t) \, dt + i \int_{t_1}^{t_2} \beta(t) \, dt \rightarrow \int_{t_1}^{t_2} z(t) \, dt
\]

\[
\begin{cases} 
\langle \bar{u}, \bar{v} \rangle, & \text{if } \int_{t_1}^{t_2} \alpha(t) \, dt \geq 0, \quad \int_{t_1}^{t_2} \beta(t) \, dt \geq 0 \\
\langle \bar{u}, \bar{v} \rangle, & \text{if } \int_{t_1}^{t_2} \alpha(t) \, dt \leq 0, \quad \int_{t_1}^{t_2} \beta(t) \, dt \leq 0 \\
-|\bar{u}|, |\bar{v}|, & \text{if } \int_{t_1}^{t_2} \alpha(t) \, dt < 0, \quad \int_{t_1}^{t_2} \beta(t) \, dt \geq 0 \\
|\bar{u}|, -|\bar{v}|, & \text{if } \int_{t_1}^{t_2} \alpha(t) \, dt \geq 0, \quad \int_{t_1}^{t_2} \beta(t) \, dt < 0
\end{cases}
\]

where

\[
\bar{u} = \sqrt{\int_{t_1}^{t_2} \alpha(t) \, dt} \sqrt{\int_{t_1}^{t_2} \alpha(t) \, dt + \int_{t_1}^{t_2} \beta(t) \, dt},
\]

\[
\bar{v} = \sqrt{\int_{t_1}^{t_2} \beta(t) \, dt} \sqrt{\int_{t_1}^{t_2} \alpha(t) \, dt + \int_{t_1}^{t_2} \beta(t) \, dt},
\]

\[
|\bar{u}| = \sqrt{\int_{t_1}^{t_2} \alpha(t) \, dt} \sqrt{\int_{t_1}^{t_2} \alpha(t) \, dt + \int_{t_1}^{t_2} \beta(t) \, dt},
\]

\[
|\bar{v}| = \sqrt{\int_{t_1}^{t_2} \beta(t) \, dt} \sqrt{\int_{t_1}^{t_2} \alpha(t) \, dt + \int_{t_1}^{t_2} \beta(t) \, dt},
\]

where \( \int_{t_1}^{t_2} z(t) \, dt \) is \( \int_{t_1}^{t_2} z(t) \, dt \) instantiated.
Chapter 5

Objective Physics

“Mathematical physics represents the purest image that the view of nature may generate in the human mind; this image presents all the character of the product of art; it begets some unity, it is true and has the quality of sublimity; this image is to physical nature what music is to the thousand noises of which the air is full ...”

- Théophile de Donder

5.0 Introduction

The previous chapters introduced ‘objective logic’, where systems are validated through instantiation. Out of objective logic emerges ‘objective mathematics’, where number systems are instantiated, including the complex numbers. Complex number instantiation involves ‘number interference’, where the real part of a complex number interferes with the imaginary part and exact values for instantiated complex numbers are not attainable except under certain circumstances.

This chapter briefly explores how objective logic can help explain the nature of the world, discusses how image theory is interpreted from a physics perspective, reviews the basic approach to applying objective logic to the laws governing the Universe and presents the results.

5.1 Objective Mathematics Interpreted

Objective mathematics is interpreted abstractly as a space that represents an observer and an external world, which exists independently of that observer. Figure 5.1-1 illustrates this relationship schematically.
5.1.1 Faithful and Unfaithful Information

If the information about an external object is in one-to-one correspondence with the nature of the object, the information is called ‘faithful’. Otherwise, it is called ‘unfaithful’.

5.1.2 The Mind, Body and the External World

Image theory adopts the premise that what the human mind perceives is the disposition of a human body and nothing else. This perception is generally called ‘self’. In other words, the ‘self’ can distinguish its mind and body from external objects, not part of itself. According to this view, the human body instantiates the human mind. Within the pairs ‘{(a),{b})’, the set of ideas ‘{b}’ is instantiated by the set ‘{-b}’, which represents the disposition of the human body. In other words, the mind contemplates a body, which is what the mind thinks it is. The pseudo-mapping ‘f({b}) = {-b}’ represents the perception the human mind has of its body. The human mind has a ‘faithful’ perception of its body if

\[ \forall b \exists f (b \in M \leftrightarrow -b \in B), \quad M \equiv \text{Mind}, \quad B \equiv \text{Body} \]

Otherwise, the human mind’s perception of its body is ‘unfaithful’.

The human body is assumed the sole source of sensual perception. The information that results from a direct observation of an external object hinges on the extent to which the external object has something in common with the disposition of the human body. As the theory goes, an external object is not directly perceptible except that this commonality exists. The ordered pairs ‘{(a),{b})’ represent the observer’s idea of an object existing in the external world, where ‘a’ signifies an awareness of the ‘external’, distinct from the observer.

An external object is represented by the pairs ‘{(-a),{b})’, where {b} represents those characteristics of the external object common to the observer’s body. The knowledge of an external object is represented by the pseudo-mapping ‘f({a}) = {-a}’, where a is instantiated by –a. Evidently, a represents information about an external object gained through some means other than sense experience. The human mind has a ‘faithful’ knowledge of an external object if

\[ \forall a \exists f (a \in M \leftrightarrow -a \in O), \quad M \equiv \text{Mind}, \quad O \equiv \text{External Object} \]

Otherwise, the human mind’s perception of an external object is ‘unfaithful’.

Those parts of an external object not empirically detectable are called ‘dark’. If a ≠ 0, b = 0, this represents an external object which is completely ‘dark’ (undetectable). It has nothing in common with the human body. Those characteristics of an external object common to an observer’s body are called ‘light’, the parts that are directly detectable.
5.1.3 What is an Observation?

An observation within image theory is represented abstractly by

\[ (a, b) \mapsto (\sqrt{a}, \sqrt{b})\sqrt{a + b}, \quad \sqrt{a}, \sqrt{b} \in R, \]

where \((\sqrt{a}, \sqrt{b})\sqrt{a + b}\) represents the information about an external object gained through observation. The quantity \(\sqrt{a}\) represents the dark part of the external object. The perceptible part is represented by the quantity \(\sqrt{b}\). If \(b = 0\), then the external object is completely undetectable. On the other hand, if \(a = 0\), \(b \neq 0\), then the external object is directly detectable.

5.1.3.1 Information Interference

The factor \(\sqrt{a + b}\) represents the instantiation of both the light- \((b)\) and the dark-parts \((a)\) of the external object. However, the factor \(\sqrt{a + b}\) symbolizes an interference, represented by the conjugate

\[ (\sqrt{a}, -\sqrt{b}) \mapsto (\sqrt{a}, \sqrt{b}) \cdot (\sqrt{a}, -\sqrt{b}) = a + b, \quad \sqrt{a}, \sqrt{b} \in R, \]

then taking its square-root \(\sqrt{a + b}\), which signifies the extent to which sense perception interferes with perfect information about the ‘dark’ parts of the external object. On the other hand, in general, an observer cannot obtain a faithful perception of the external object to the extent that sense perception is interfered with by the dark-parts of the external object. This is represented by the anti-conjugate

\[ (-\sqrt{a}, \sqrt{b}) \mapsto (\sqrt{a}, \sqrt{b}) \cdot (-\sqrt{a}, \sqrt{b}) = -(a + b), \quad \sqrt{a}, \sqrt{b} \in R, \]

then taking its square-root \(i\sqrt{a + b}\), which represents an interference by the non-empirical parts of the external object.

5.1.4 A Summary of Objective Mathematics Interpreted

As a matter of general process, problems in image theory begin by describing the physics of an external object represented by \(\langle a, b \rangle\), where \(a\) and \(b\) abstractly symbolize the dark- and light-parts of the object respectfully. In this case, the independent world, where the object resides, is hidden. When an observation is made, the physics is described by mathematics in instantiation space, where number interference applies. This is similar to what happens in quantum mechanics, where the knowledge about a system is represented by a ‘wave function’. The wave function represents the possible states that the system can adopt. But once an observation is made, the wave function ceases to be a factor and only one of the possibilities materializes.
5.2 Mathematical Preliminaries – the Fundamental Function

The previous chapter introduced the function

\[ f(t) = \left(-\frac{1}{G}\right)^{1-t}G^t, \quad G \in R \]

This function is called ‘the fundamental’ function of image theory. The domain of \( f \) is generally all positive real numbers and zero i.e. \( t \geq 0 \in R \). This restriction is not mathematically necessary, but physically, \( t \) will, in general, represent ‘time’, which is normally understood as unidirectional. However, the range of \( f \) is a set of complex numbers. Geometrically, \( f \) draws out a space curve in a two-dimensional complex space, where \( t \) is the ‘parameter’. Note that if \( G \) is replaced by \(-G\), then

\[ f(t) = \left(\frac{1}{G}\right)^{1-t}(-G)^t \]

The substitution only serves to draw out the space curve, whatever it might be, in the opposite direction. For instance, if \( G = 1 \) or \( G = -1 \), \( f \) produces a complex unit circle with period ‘2’ (see fig. 5.2-1). For instance, if \( G = -1 \), then

\[ f(t) = (-1)^t = \cos \pi t + i \sin \pi t = e^{i\pi t} \]

![Figure 5.2-1](image)

If \( G = -\sqrt{10} e^{1/2}/10 \), then \( f \) produces a complex spiral (see fig. 5.2-2):

\[ f(t) = 10^{1/2-t}e^{-1/2+t}(\cos \pi t + i \sin \pi t) = A(t)e^{i\pi t}, \quad A(t) = 10^{1/2-t}e^{-1/2+t} \]
5.2.1 The Exponential Inverse of the Fundamental Function

Note that $f$ has an 'exponential inverse', designated '$g$':

$$g(t) = \left(-\frac{1}{G}\right)^{t-1} (G)^{-t} \to fg = 1$$

In image theory, $f$ will be associated with large-scale physics and $g$ with small-scale physics. If $G = -1$, then the locus of $g$ is a complex unit circle i.e.

$$g(t) = \cos \pi t - i \sin \pi t = e^{-i\pi t}$$

And if $G = -\sqrt{10} e^{1/2}/10$, for example, then the locus of $g$ is a complex spiral on a very large scale (see fig. 5.2.1-1):

$$g(t) = 10^{(-1/2+t)}e^{(1/2-t)}(\cos \pi t - i \sin \pi t) = A(t)e^{-i\pi t}, \quad A(t) = 10^{(-1/2+t)}e^{(1/2-t)}$$
5.2.2 The Square Root of the Fundamental Function

Evidently, if $G \to \sqrt{G}$, then $f \to \sqrt{f}$. Note that

$$\left(\frac{1}{G}\right)^{1-t}G^t \to \left(\frac{1}{\sqrt{G}}\right)^{1-t}G^\frac{1}{2} = \left(\left(\frac{1}{G}\right)^{1-t}\right)\left(G^\frac{1}{2}\right) = \left(\frac{1}{\sqrt{G}}\right)^{1-t}(\sqrt{G})^t, \quad G > 0 \in \mathbb{R}, \quad t \geq 0 \in \mathbb{R}$$

To see this,

$$f = \left(\frac{1}{G}\right)^{1-t}G^t = \left(\frac{1}{G}\right)^{1-1}G^t = \frac{1}{G}G^{2t}$$

Define

$$\sqrt{f} = \frac{1}{\sqrt{G}}G^t,$$

since

$$\frac{1}{\sqrt{G}}\frac{1}{\sqrt{G}}G^tG^t = \frac{1}{G}G^{2t} = f, \quad G > 0 \in \mathbb{R}$$

Hence, $\sqrt{f(t)}$ plays the role of a square root function. For each $t$, $G^t/\sqrt{G}$ produces the square root of $f(t)$.

However, a minus sign in front of $1/G$ makes $f$ complex. Note that

$$\left(\frac{-1}{G}\right)^{1-t}G^t \not= \left(\frac{-1}{\sqrt{G}}\right)^{1-t}(\sqrt{G})^t$$

It would be desirable to find an 's' such that

$$\left[\left(\frac{-1}{G}\right)^{s1-t}\right]G^\frac{1}{2} = \left(\frac{-1}{\sqrt{G}}\right)^{1-t}(\sqrt{G})^t$$

Let

$$\left(\frac{-1}{G}\right)^s = \frac{-1}{\sqrt{G}} \to s = \frac{2i\pi - \ln(G)}{2\ln\left(-\frac{1}{G}\right)} = \frac{i\pi - \ln(\sqrt{G})}{\ln(-1/G)}$$

$$f = \left(\frac{-1}{G}\right)^{1-t}G^t \to \left[\left(\frac{-1}{G}\right)^s\right]^{1-t}G^\frac{1}{2}$$

$$= \left(\frac{-1}{\sqrt{G}}\right)^{1-t}(\sqrt{G})^t.$$
where
\[
\left(\frac{-1}{\sqrt{G}}\right)^{1-t} (\sqrt{G})^t
\]
plays the role of the square root function of \(f\). In other words, define
\[
\sqrt{f} = \left(\frac{-1}{G}\right)^s G^z = \left(\frac{-1}{\sqrt{G}}\right)^{1-t} (\sqrt{G})^t
\]
Note that
\[
\left(\frac{-1}{G}\right)^s G^z \neq \left(\frac{-1}{G}\right)^{s(1-t)},
\]
since \(f\) is complex.

**5.2.3 Differentiating the Fundamental Function**

If
\[
f(t) = \left(\frac{-1}{G}\right)^{1-t} G^t,
\]
then
\[
\frac{df}{dt} = -\frac{1}{G} \frac{d}{dt} \left[\left(\frac{-1}{G}\right)^{-t} G^t\right] = -\frac{1}{G} \left[\left(-\frac{1}{G}\right)^{-t} \ln \left(-\frac{1}{G}\right) G^t + \left(-\frac{1}{G}\right)^{-t} G^t \ln G\right] = -\left[\ln \left(-\frac{1}{G}\right) - \ln G\right] \left(-\frac{1}{G}\right)^{1-t} G^t \rightarrow \frac{df}{dt} = -\lambda f, \quad \lambda = \ln \left(-\frac{1}{G}\right) - \ln G
\]

Note that
\[
\frac{d^2 f}{dt^2} = \left[\ln \left(-\frac{1}{G}\right) - \ln G\right]^2 \left(-\frac{1}{G}\right)^{1-t} G^t
\]
In general,
\[
\frac{d^nf}{dt^n} = (-1)^n \left[\ln \left(-\frac{1}{G}\right) - \ln G\right]^n \left(-\frac{1}{G}\right)^{1-t} G^t \rightarrow \frac{d^nf}{dt^n} = (-\lambda)^n f, \quad \lambda = \ln \left(-\frac{1}{G}\right) - \ln G,
\]

**5.2.3.1 Differentiating the Inverse of the Fundamental Function**

Moreover, if
\[
g(t) = \left(\frac{-1}{G}\right)^{t-1} G^{-t},
\]
\[97\]
\[
\frac{dg}{dt} = \left(-\frac{1}{G}\right)^{-1} \frac{d}{dt} \left[\left(-\frac{1}{G}\right)^t G^{-t}\right] = \left(-\frac{1}{G}\right)^{-1} \left[\left(-\frac{1}{G}\right)^t \ln \left(-\frac{1}{G}\right)G^{-t} - \left(-\frac{1}{G}\right)^t G^{-t} \ln G\right]
\]
\[
= \left[\ln \left(-\frac{1}{G}\right) - \ln G\right] \left(-\frac{1}{G}\right)^{t-1} G^{-t} \to \frac{dg}{dt} = \lambda g, \quad \lambda = \ln \left(-\frac{1}{G}\right) - \ln G
\]

Note that
\[
\frac{d^2 g}{dt^2} = \left[\ln \left(-\frac{1}{G}\right) - \ln G\right]^2 \left(-\frac{1}{G}\right)^{t-1} G^{-t}
\]

In general,
\[
\frac{d^n g}{dt^n} = \left[\ln \left(-\frac{1}{G}\right) - \ln G\right]^n \left(-\frac{1}{G}\right)^{t-1} G^{-t} \to \frac{d^n g}{dx^n} = \lambda^n g, \quad \lambda = \ln \left(-\frac{1}{G}\right) - \ln G, \quad n \in N
\]

### 5.2.4 Integrating the Fundamental Function

Let
\[
\int f(t) \, dt = \int \left(-\frac{1}{G}\right)^{1-t} G^t \, dt
\]

Integrating by parts, let
\[
u = G^t \to dv = G^t \ln G \, dt, \quad dv = G^t \ln G \, dt,
\]
then
\[
\int f(t) \, dt \to \int u \, dv = uv - \int v \, du \to \int \left(-\frac{1}{G}\right)^{1-t} G^t \ln G \, dt
\]
\[
= \left(-\frac{1}{G}\right)^{1-t} G^t + \int \left(-\frac{1}{G}\right)^{1-t} \ln \left(-\frac{1}{G}\right)G^t \, dt
\]
\[
\to \int \left(-\frac{1}{G}\right)^{1-t} G^t \ln G \, dt - \int \left(-\frac{1}{G}\right)^{1-t} \ln \left(-\frac{1}{G}\right)G^t \, dt = \left(-\frac{1}{G}\right)^{1-t} G^t
\]
\[
\to \left[\ln G - \ln \left(-\frac{1}{G}\right)\right] \int \left(-\frac{1}{G}\right)^{1-t} G^t \, dt = \left(-\frac{1}{G}\right)^{1-t} G^t
\]

Hence,
\[
\int f(t) \, dt = \int \left(-\frac{1}{G}\right)^{1-t} G^t \, dt = -\frac{\left(-\frac{1}{G}\right)^{1-t} G^t}{\ln \left(-\frac{1}{G}\right) - \ln G} = -\frac{1}{\lambda} f, \quad \lambda = \ln \left(-\frac{1}{G}\right) - \ln G
\]
Moreover,
\[
\int \left( \int f(t) \, dt \right) \, dt = \frac{(-\frac{1}{G})^{1-t} G^t}{\ln\left(-\frac{1}{G}\right) - \ln G} = \left(-\frac{1}{\lambda}\right)^2 f
\]

In general,
\[
\int \ldots \int f(t) \, d^n t = (-1)^n \frac{(-\frac{1}{G})^{1-t} G^t}{\ln\left(-\frac{1}{G}\right) - \ln G}^n = \left(-\frac{1}{\lambda}\right)^n f, \quad n \in \mathbb{N}
\]

### 5.2.4.1 Integrating the Exponential Inverse of the Fundamental Function

Furthermore, if
\[
g(t) = \left(-\frac{1}{G}\right)^{t-1} G^{-t} \rightarrow \int g(t) \, dt = \frac{(-\frac{1}{G})^{t-1} G^{-t}}{\ln\left(-\frac{1}{G}\right) - \ln G}
\]
then
\[
\int \left( \int g(t) \, dt \right) \, dt = \frac{\left(-\frac{1}{G}\right)^{t-1} G^{-t}}{\ln\left(-\frac{1}{G}\right) - \ln G}^2
\]

In general,
\[
\int \ldots \int g(t) \, d^n t = \frac{(-\frac{1}{G})^{t-1} G^{-t}}{\ln\left(-\frac{1}{G}\right) - \ln G}^n = \left(\frac{1}{\lambda}\right)^n g, \quad \lambda = \ln\left(-\frac{1}{G}\right) - \ln G, \quad n \in \mathbb{N}
\]

### 5.2.5 Relationship between the Derivative and Integral

Note that
\[
\frac{df}{dt} \int f(t) \, dt = -\lambda \left(-\frac{1}{\lambda}\right) f^2 = f^2
\]
In general,
\[
\frac{d^n f}{dt^n} \int \ldots \int f(t) \, d^n t = (-\lambda)^n f \left(-\frac{1}{\lambda}\right)^n f = f^2, \quad n \in \mathbb{N}
\]
So evidently,

\[
\sqrt{\frac{d^n f}{dt^n}} \int \ldots \int f(t) \, d^n t = \frac{d^n}{dt^n} \left( \int \ldots \int f(t) \, d^n t \right)
\]

Similarly,

\[
\frac{d^n g}{dt^n} \int \ldots \int g(t) \, d^n t = \left( \frac{1}{\lambda} \right)^n g \lambda^n g = g^2 \rightarrow \sqrt{\frac{d^n g}{dt^n}} \int \ldots \int g(t) \, d^n t = \frac{d^n}{dt^n} \left( \int \ldots \int g(t) \, d^n t \right)
\]

5.3 The Physics of Image Theory

Recall that

\[ e = \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n, \quad n \in \mathbb{N} \]

Remarkably, there appears to be a meaningful physical relationship between the irrational number ‘e’ and the rational number ‘10’. On the face of it, such a claim seems unfounded. But consider

\[ 10^{15} \cdot e = e \rightarrow G = \frac{e}{10^{15}} \rightarrow \sqrt{G} = \pm \frac{e}{\sqrt{10^{15}}} \approx \pm 5.213714442 \times 10^{-8} \]

Numerically, \(|\sqrt{G}|\) is approximately equal to the universal constant associated with the Newtonian theory of gravity in the ‘CGS’ system of units ([cm – gm – sec]). The experimentally measured value of Newton’s constant is

\[ \approx 6.67 \times 10^{-8} \text{[cm}^3/\text{gm} \cdot \text{sec}^2] \]

The difference between the numerical and experimental values is \(\approx 10^{-8}\).

5.3.1 The Principle of Equivalence within Image Theory

In Newton’s as well as Einstein’s theory of gravity, gravitational mass is equivalent to inertial mass, which implies that all massive objects in a uniform gravitation field accelerate at the same rate. Numerous experiments have validated the equivalence principle. In Newton’s formulation, the equivalence of inertial with gravitational mass is expressed

\[ F = ma = \frac{GMm}{r^2} \rightarrow a = \frac{GM}{r^2}, \]

where \(F\) represents the force due to gravity, \(a\) acceleration, \(G\) Newton’s constant, \(M, m\), the masses of the bodies and \(r\) is the distance between the centers of the two masses. In other words, the acceleration ‘\(a\)’ is independent of \(m\).
Let \( a \equiv \frac{d^2S}{dt^2} \), where \( S \) represents ‘distance’ and \( t \) ‘time’. According to image theory, \( a \) has a dark- and a light-part. Let

\[
a_D + i a_L = \left( \frac{d^2S}{dt^2} \right)_D + i \left( \frac{d^2S}{dt^2} \right)_L,
\]

where \( \left( \frac{d^2S}{dt^2} \right)_D \) represents the dark- and \( \left( \frac{d^2S}{dt^2} \right)_L \) the light-part – the part that is directly detectable.

Within image theory, an observation is represented abstractly by projection into instantiation space i.e.

\[
\langle a_D, a_L \rangle \rightarrow \langle \sqrt{a_D}, \sqrt{a_L} \rangle \sqrt{a_D + a_L}
\]

Projection into instantiation space requires finding the square roots of \( a_D \) and \( a_L \). Hence, define

\[
\sqrt{a_D} + i \sqrt{a_L} = \sqrt{\left( \frac{d^2S}{dt^2} \right)_D} + i \sqrt{\left( \frac{d^2S}{dt^2} \right)_L}
\]

and let

\[
a_D + i a_L = \left( \frac{d^2S}{dt^2} \right)_D + i \left( \frac{d^2S}{dt^2} \right)_L \propto \left( -\frac{1}{G} \right)^{1-t} \sqrt{G} = f \rightarrow \sqrt{\left( \frac{d^2S}{dt^2} \right)_D} + i \sqrt{\left( \frac{d^2S}{dt^2} \right)_L} \propto \sqrt{f}
\]

where the expression on the right-side is the fundamental function of image theory. Note, in this case, acceleration is a function of time.

A sensible physics equation requires that if the parameter \( 't' \) represents ‘time’, measured in seconds, a quantity ‘\( \nu \)’ with dimension \([1/sec]\) must be introduced so that the product \( \nu t \) is dimensionless. This avoids dimensions in the exponent and suggests that \( \nu \) represents a ‘vibration’ ([cycles per sec]), the nature of which is unknown at this point. If \( \nu \) is intended to measure the frequency of time, then it will be convenient to set \( \nu = 1 \), where the cycles that measure time are in phase with time itself. The relation above then becomes

\[
\sqrt{\left( \frac{d^2S}{dt^2} \right)_D} + i \sqrt{\left( \frac{d^2S}{dt^2} \right)_L} \propto \left( \frac{1}{\sqrt{G}} \right)^{1-\nu t} \left( \sqrt{G} \right)^\nu, \quad \nu = 1
\]

The units on the left-hand side are \([\sqrt{cm/sec^2}]\). Introducing units \([k] = 1[cm/sec^2]\) as a factor on the right-hand side makes the dimensions on each side of the relation equal.
Here $\sqrt{G}$ is treated as a dimensionless constant. The relation then becomes the equation

$$\sqrt{\left(\frac{d^2S}{dt^2}\right)_D} + i \sqrt{\left(\frac{d^2S}{dt^2}\right)_L} = [\sqrt{k}] \left(- \frac{1}{\sqrt{G}}\right)^{1-\nu t} (\sqrt{G})^\nu$$

### 5.3.2 Deriving the Speed of Light

Let

$$[\sqrt{k}] \int \left(- \frac{1}{\sqrt{G}}\right)^{1-\omega} (\sqrt{G})^\omega d\omega = \left[ \frac{k}{\sqrt{v}} \right] \int \left(- \frac{1}{\sqrt{G}}\right)^{1-\omega} (\sqrt{G})^\omega d\omega, \quad \omega = vt,$$

where $\omega$ is dimensionless and $\sqrt{v}$ has units $[1/\text{sec}]$. Hence,

$$\sqrt{\int \left(\frac{d^2S}{dt^2}\right)_D dt} + i \sqrt{\int \left(\frac{d^2S}{dt^2}\right)_L dt} \rightarrow \sqrt{\left(\frac{dS}{dt}\right)_D} + i \sqrt{\left(\frac{dS}{dt}\right)_L} = \left[ \frac{k}{\sqrt{v}} \right] \int \left(- \frac{1}{\sqrt{G}}\right)^{1-\omega} (\sqrt{G})^\omega d\omega$$

$$= \left[ \frac{k}{\sqrt{v}} \right] \frac{2 \left(- \frac{1}{\sqrt{G}}\right)^{1-\omega} (\sqrt{G})^\omega}{\ln \left(- \frac{1}{\sqrt{G}}\right) + \ln G} + C = \left[ \frac{k}{\sqrt{v}} \right] \frac{\left(- \frac{1}{\sqrt{G}}\right)^{1-\omega} (\sqrt{G})^\omega}{\ln \left(- \frac{1}{\sqrt{G}}\right) + \ln G} + C, \quad \omega = vt$$

Let $\omega$ vary from 0 to $\infty$, then by direct calculation

$$\sqrt{\left(\frac{dS}{dt}\right)_D} + i \sqrt{\left(\frac{dS}{dt}\right)_L} = \left[ \frac{cm}{\sqrt{\text{sec}}} \right] \left(- \frac{1}{\sqrt{G}}\right)^{1-\omega} (\sqrt{G})^\omega \bigg|_{0}^{\infty}$$

$$\approx (-5.669067105 \times 10^5 + i53102.41304) \left[ \sqrt{\text{cm/sec}} \right]$$

Note that the dimension associated with this result is the square-root of a velocity.

Define

$$\sqrt{x} = 5.669067105 \times 10^5, \quad \sqrt{y} = 53102.41304$$

When these numbers are projected into instantiation space i.e. $(-\sqrt{x}, \sqrt{y})\sqrt{x + y}$, then the left-side of the equation becomes
\[ \langle \sqrt{\left(\frac{dS}{dt}\right)_D}, \sqrt{\left(\frac{dS}{dt}\right)_L} \rangle \sqrt{\left(\frac{dS}{dt}\right)_D + \left(\frac{dS}{dt}\right)_L} = \langle \sqrt{\left(\frac{dS}{dt}\right)_D \left(\frac{dS}{dt}\right)_D} + \left(\frac{dS}{dt}\right)_D \left(\frac{dS}{dt}\right)_L, \sqrt{\left(\frac{dS}{dt}\right)_D \left(\frac{dS}{dt}\right)_L} + \left(\frac{dS}{dt}\right)_L \left(\frac{dS}{dt}\right)_L \rangle \]

Let
\[ \left(\frac{dS}{dt}\right)_D = \sqrt{\left(\frac{dS}{dt}\right)_D \left(\frac{dS}{dt}\right)_D} + \left(\frac{dS}{dt}\right)_D \left(\frac{dS}{dt}\right)_L, \quad \left(\frac{dS}{dt}\right)_L = \sqrt{\left(\frac{dS}{dt}\right)_D \left(\frac{dS}{dt}\right)_L} + \left(\frac{dS}{dt}\right)_L \left(\frac{dS}{dt}\right)_L, \]

then
\[ \left(\frac{dS}{dt}\right)_D + i \left(\frac{dS}{dt}\right)_L = (\sqrt{x} + i\sqrt{y})\sqrt{x + y} \]
\[ \approx (-3.227900723 \times 10^{11} + i3.023589495 \times 10^{10}) \text{ [cm/sec]} \]

In particular,
\[ \left(\frac{dS}{dt}\right)_L \approx 3.023589495 \times 10^{10} \text{ [cm/sec]}, \]

which appears to be approximately the detectable speed of light. The measured speed of detectable light is 2.99792458 \times 10^{10} \text{ [cm/sec]}. However,
\[ 3.023589495 \times 10^{10} - 2.99792458 \times 10^{10} = 2.5664915 \times 10^{8} \text{ [cm/sec]}, \]
a dramatic difference between the two speeds. However, the discrepancy is likely not as significant as it first might appear as explained by the following line of argument.

Suppose the speed of light 'c' is constant and that its speed is 3. At time 't = 0', \( c = 3 \). And at some later time, say 't = 13', if \( c = 3 \), then the average speed \( c_{ave} \) is
\[ c_{ave} = \frac{13 \times 3}{13} = 3, \]

as expected. On the other hand, if the speed of light is not constant, but has a limit, as image theory claims, then, say, for example, in the first second, \( c = 1 \) (on average) and, in the next two seconds, \( c = 2 \) (on average) and, finally, in the final ten seconds, the speed of light travels at its limit speed '3'. In this case,
\[ c_{ave} = \frac{1 \times 1 + 2 \times 2 + 10 \times 3}{13} \approx 2.7 < 3 \]

What must the limit speed be if \( c_{ave} \) is to equal 3? It follows that
\[
\frac{(1 \times 1) + (2 \times 2) + 10x}{13} = 3 \rightarrow x = 3.4,
\]

where \(x\) is the ‘limit speed’. Note that the limit speed must be greater than 3 in order that the average speed computes to 3. This can explain why the limit speed of light is slightly greater than the measured speed of light, assumed constant.

Note that
\[
\left(\frac{d\bar{S}}{dt}\right)_D \approx -3.227900723 \times 10^{11} \text{ [cm/sec]}
\]

Evidently, \((d\bar{S}/dt)_D\) is that part of the light wave that is undetectable (dark).

### 5.3.2.1 The Difference Between Dark- and Light-Light Limit Speeds

Let
\[
c = 3.023589495 \times 10^{10}, \quad c' = -3.227900723 \times 10^{11},
\]

then
\[
i \left(\frac{d\bar{S}}{dt}\right)_L = ic \rightarrow - \left(\frac{d\bar{S}}{dt}\right)_L^2 = -c^2, \quad \frac{d\bar{S}}{dt}_D \Rightarrow \left(\frac{d\bar{S}}{dt}\right)_D^2 = c'^2
\]

Now,
\[
\left(\frac{d\bar{S}}{dt}\right)_D^2 - \left(\frac{d\bar{S}}{dt}\right)_L^2 = v_D^2 - v_L^2 = c'^2 - c^2 \rightarrow \sqrt{v_D^2 - v_L^2} = \sqrt{c'^2 - c^2}
\]
\[
\approx 3.213708473 \times 10^{11} \text{ [cm/sec]}, \quad \left(\frac{d\bar{S}}{dt}\right)_D^2 = v_D^2, \quad \left(\frac{d\bar{S}}{dt}\right)_L^2 = v_L^2
\]

The expression \(\sqrt{c'^2 - c^2}\) represents the difference between the limit speeds of the \(D\)-light and the \(L\)-light. Image theory predicts dark-light.

### 5.3.3 Inverse Acceleration

In the previous section, physics on the large-scale was briefly examined in objective space, where image theory predicts ‘dark-light’. On the small scale, the only option is to employ the exponential inverse of the fundamental function. Let
\[
\sqrt{\left(\frac{dt^2}{d^2S}\right)_D} + i \sqrt{\left(\frac{dt^2}{d^2S}\right)_L} = \left[\sqrt{1/k}\right] \sqrt{\frac{1}{\left(-\frac{1}{G}\right)^{1-\omega} G^{\omega}}} = \left[\sqrt{1/k}\right] \sqrt{G} \cdot \omega = vt
\]
\[
= \left[\sqrt{1/k}\right] \left(-\frac{1}{\sqrt{G}}\right)^{\omega-1} \left(\sqrt{G}\right)^{-\omega}, \quad t = 1, \quad \omega = vt
\]
Note that \([1/k] = 1[sec^2/cm]\) and that the expression on the right-side is simply the exponential inverse of the square root of the fundamental function. Whatever

\[
\sqrt{\frac{dt^2}{d^2S}}_D + i \sqrt{\frac{dt^2}{d^2S}}_L
\]

represents, the quantities under the radical signs are functions of \(\omega\). Let

\[
\left[\sqrt{1/kt}\right] \left(\int \frac{1}{\alpha(\omega)} D\omega + i \int \frac{1}{\beta(\omega)} L\omega\right) = \left[\sqrt{1/(kt)}\right] \int \left(-\frac{1}{\sqrt{G}}\right)^{\omega-1} (\sqrt{G})^{-\omega} d\omega
\]

\[
= \left[\sqrt{1/(kt)}\right] \frac{2}{2\ln\left(-\frac{1}{\sqrt{G}}\right) - \ln G} - C = \left[\sqrt{1/(kt)}\right] \frac{\left(-\frac{1}{\sqrt{G}}\right)^{\omega-1}}{\ln\left(-\frac{1}{\sqrt{G}}\right) - \ln \sqrt{G}} + C,
\]

\(\omega = vt, \quad [k] = 1[cm/\text{sec}^2]\)

In the derivation above, it is assumed that \(v\) varies and \(t\) is held constant. To find the limits of integration for \(\int \sqrt{g} \, d\omega\), start with

\[
\int_0^\infty \sqrt{f} \, d\omega
\]

The upper limit is inverted to find the lower limit of \(\int \sqrt{g} \, d\omega\) i.e.

\[
\lim_{\omega \to \infty} 1/\omega = 0
\]

Likewise, the lower limit is inverted to find the upper limit, which becomes \(1/0\), an undefined quantity. One approach to overcoming the problem of the undefined quantity is to argue that \(\lim_{\omega \to 0} 1/\omega = \infty\). But, by direct calculation

\[
\left(\frac{-1}{\sqrt{G}}\right)^{\omega-1} (\sqrt{G})^{-\omega} \bigg|_0^\infty = \infty
\]

Infinity, in this context, is a nonsensical answer.

5.3.3.1 Renormalizing the Inverse of the Fundamental Function

Effectively, renormalization is the only approach to dealing with the infinity problem. Renormalization requires specifying a cutoff value that makes a sensible theoretical prediction and, at the same time, gets rid of the infinity. Let
\[
\sqrt{\int \left( \frac{1}{\alpha(\omega)} \right)_D \, d\omega} + i \sqrt{\int \left( \frac{1}{\beta(\omega)} \right)_L \, d\omega} = \left( -\frac{1}{\sqrt{G}} \right)^{\frac{\omega-1}{2}} \left( \frac{\sqrt{G}}{\ln \left( -\frac{1}{\sqrt{G}} \right) - \ln \sqrt{G}} \right)|_0^a = ?,
\]

where \( a \) represents the 'cutoff' value.

**5.3.3.2 Finding a Cutoff**

A somewhat sensible argument for finding a cutoff can be made along the following lines. Recall that

\[
fg = 1 \rightarrow \int fg \, d\omega = \int d\omega \rightarrow \int f \, d\omega = K,
\]

where \( K \neq 0 \in R \) represents an arbitrary real number.

Suppose that

\[
\left( \int f \, d\omega \right) \left( \int g \, d\omega \right) = K, \quad K \in R,
\]

which is not to be taken seriously, since the equation is not, in general, true. Multiplying two complex numbers together generally produces a complex number. However, the approach is only intended as an approximation, so the fact that the procedure lacks technical correctness is, at this point, not important. Further, suppose

\[
\left( \int f \, d\omega \right) \left( \int d\omega \right) = \frac{K}{g} = Kf \rightarrow \int d\omega = \frac{Kf}{\int f \, d\omega}, \quad f = \frac{1}{g}
\]

Taking the definite integral of the denominator on the right-side leaves

\[
\int d\omega = \frac{Kf}{\int_0^\infty f \, d\omega}
\]

From a previous calculation,

\[
\frac{1}{\int_0^\infty f \, d\omega} = x' + iy' \in C
\]

Hence,

\[
\int d\omega = \frac{Kf}{\int_0^\infty f \, d\omega} = K(x' + iy')f \rightarrow \frac{1}{K} \int d\omega = (x' + iy')f = kf, \quad k = x' + iy'
\]

106
Evidently, then
\[
\frac{1}{K} \int_{0}^{\omega} d\omega = \frac{1}{K} \omega = kf(\omega) - kf(0) = x + iy, \quad kf(\omega) - kf(0) \in \mathbb{C}
\]

The equation above only makes sense if \( y \approx 0 \), since \( \omega, K \in \mathbb{R} \). This suggests that
\[
\frac{1}{K} \omega \approx x
\]

According to image theory, each side of the equation \( \omega = x + iy \), \( y = \varepsilon > 0 \ll 1 \), must be instantiated. The instantiation of \( \omega \) gives
\[
(\sqrt{x}, \sqrt{y})\sqrt{x + y} = (\sqrt{x}, \sqrt{x + y}) \rightarrow (\sqrt{x})^2 = x = x^2 + xy,
\]
\[
(i \sqrt{y})^2 = -y(x + y) = -y^2 - xy
\]

Hence,
\[
(x^2 - y^2) + (i \sqrt{y})^2 = x^2 - y^2 \rightarrow \omega \approx \sqrt{x^2 - y^2} \in \mathbb{R}
\]

The fact that the derivation above is a bit mathematically dubious is not too important. The procedure is only intended to provide a reasonable estimate for a cutoff value. The task, then, is to search for a value \( 'a' \) such that
\[
\frac{1}{K} \int_{0}^{a} g \, d\omega \approx a + ib, \quad b \ll 1,
\]

where \( a + ib \) is a yet known complex number.

Specifically,
\[
\left( \int_{0}^{\infty} \sqrt{\left(-\frac{1}{G}\right)^{1-\omega}} G^{\omega} \, d\omega \right) \left( \int_{0}^{a} \sqrt{\left(-\frac{1}{G}\right)^{-1+\omega}} G^{-\omega} \, d\omega \right)
\]
\[
= \left( \int_{0}^{\infty} \left(-\frac{1}{\sqrt{G}}\right)^{1-\omega} (\sqrt{G})^{\omega} \, d\omega \right) \left( \int_{0}^{a} \left(-\frac{1}{\sqrt{G}}\right)^{-1} (\sqrt{G})^{-\omega} \, d\omega \right) = \sqrt{K},
\]

where \( \sqrt{K} \) is a dimensionless constant. Now
\[
\int_{0}^{a} \left(-\frac{1}{\sqrt{G}}\right)^{\omega-1} (\sqrt{G})^{-\omega} \, d\omega = \frac{\sqrt{K}}{\int_{0}^{\infty} \left(-\frac{1}{\sqrt{G}}\right)^{1-\omega} (\sqrt{G})^{\omega} \, d\omega
where $a$ is the 'cutoff'. But, by a previous calculation, the dominator on the right-side of the equation above is computable. In fact,

\[
\int_0^a \left( -\frac{1}{\sqrt{G}} \right)^{\omega-1} (\sqrt{G})^{-\omega} d\omega \approx \frac{\sqrt{K}}{\sqrt{x} + i\sqrt{y}}
\]

where $\sqrt{x} = -5.669067105 \times 10^5$ and $\sqrt{y} = 53102.41304$. As an approximation, let

\[
\int_0^a \left( -\frac{1}{G} \right)^{1+\omega} G^{-\omega} d\omega \approx \left( \frac{\sqrt{K}}{\sqrt{x} + i\sqrt{y}} \right)^2 = \left( \frac{\sqrt{K}}{\sqrt{x} + i\sqrt{y}} \right) \left( \frac{\sqrt{K}}{\sqrt{x} + i\sqrt{y}} \right)
\]

By direct calculation

\[
z = \int_0^a \left( -\frac{1}{G} \right)^{1+\omega} G^{-\omega} d\omega \approx K(3.030829651 \times 10^{-12} + i5.728244736 \times 10^{-13})
\]

\[
\Longrightarrow \frac{1}{K} \int_0^a \left( -\frac{1}{G} \right)^{1+\omega} G^{-\omega} d\omega
\]

\[
= 3.030829651 \times 10^{-12} + i5.728244736 \times 10^{-13}
\]

Now let $\sqrt{x} = \sqrt{K}3.030829651 \times 10^{-12}$ and $\sqrt{y} = \sqrt{K}5.728244736 \times 10^{-13}$, then

\[
\frac{\bar{Z}}{K} = (\sqrt{x} + i\sqrt{y})\sqrt{x + y} \approx (3.304854273 \times 10^{-12} + i1.436753728 \times 10^{-12})
\]

Moreover,

\[
(\sqrt{x}\sqrt{x + y})^2 = (3.304854273 \times 10^{-12})^2,
\]

\[
(i\sqrt{y}\sqrt{x + y})^2 = -(1.436753728 \times 10^{-12})^2 \rightarrow x(x + y) - y(x + y)
\]

\[
= x^2 - y^2 = 8.857800495 \times 10^{-24} \rightarrow \sqrt{x^2 - y^2} \approx 2.976205721 \times 10^{-12}
\]

Recall that

\[
\frac{1}{K} \int_0^a \left( -\frac{1}{G} \right)^{1+\omega} G^{-\omega} d\omega = (3.030829651 \times 10^{-12} + i5.728244736 \times 10^{-13})
\]

and note that

\[
2.976205721 \times 10^{-12} \approx 3.030829651 \times 10^{-12}
\]

Hence, a hypothesis is made that

\[
a \approx 3.030829651 \times 10^{-12},
\]

where $a$ is the cutoff.
5.3.3.3 Specifying the Cutoff

Specifically, let

\[ a = v_{\text{cutoff}} \approx \frac{1}{\sqrt{c'^2 - c^2}} \]

At this point, a case for this assertion is not supportable. For one thing, \( a = vt \) implies that \( v \) has dimensions \('[1/T]'\), while \( 1/\sqrt{c'^2 - c^2} \) has dimensions \('[T/L]'\). In order to obtain the correct dimensions, \( v_{\text{cutoff}} \) must be multiplied by a factor \('[L/T^2]'\), the dimensions for acceleration. This suggests that the problem at hand is missing an acceleration factor, which, at this point, is unknown.

Let

\[ v_{\text{cutoff}} = \sqrt{\frac{1}{(c'^2 - c^2)}} = \frac{1}{3.213708473 \times 10^{11}} \approx 3.111669925 \times 10^{-12} \rightarrow a = v_{\text{cutoff}} t, \]

\[ t = 1, \]

then by direct calculation

\[
\left[ \sqrt{1/(kt)} \right] \int_0^a \left( -\frac{1}{G} \right)^{-1+\omega} G^{-\omega} d\omega = \left[ \sqrt{1/(kt)} \right] \left( \frac{-1}{\sqrt{G}} \right)^{\omega-1} \left( \sqrt{G} \right)^{-\omega} \left[ \frac{\ln \left( \frac{-1}{\sqrt{G}} \right) - \ln \sqrt{G} }{0} \right]
\]

\[ \approx (-1.622335843 \times 10^{-19} - i7.956265238 \times 10^{-31}) \left[ \text{sec/cm} \right] \]

Note that

\[-1.622335843 \times 10^{-19} - i7.956265238 \times 10^{-31} \]

\[ = -\sqrt{G}(3.111669925 \times 10^{-12} + i1.526026276 \times 10^{-23}) \]

\[ = -\frac{1}{\sqrt{G}} \int_0^a \left( -\frac{1}{G} \right)^{-1+\omega} G^{-\omega} d\omega = \left( \frac{-1}{\sqrt{G}} \right)^{\omega-1} \left( \sqrt{G} \right)^{-\omega} \left[ \frac{\ln \left( \frac{-1}{\sqrt{G}} \right) - \ln \sqrt{G} }{0} \right]
\]

\[ = (3.111669925 \times 10^{-12} + i1.526026276 \times 10^{-23}) \left[ \text{sec/cm} \right] \]

Now let \( \sqrt{x} = 3.111669925 \times 10^{-12} \) and \( \sqrt{y} = 1.526026276 \times 10^{-23} \), then let

\[ \bar{z} = (\sqrt{x} + i\sqrt{y})\sqrt{x + y} \approx (9.682489722 \times 10^{-24} + i4.748490068 \times 10^{-35}) \left[ \text{sec/cm} \right] \]
Note that

\[ (\sqrt{x \sqrt{x + y}})^2 = (9.682489722 \times 10^{-24})^2, \]
\[ (i \sqrt{y \sqrt{x + y}})^2 = -(4.748490068 \times 10^{-25})^2 \rightarrow \sqrt{x(x + y) - y(x + y)} \]
\[ = \sqrt{x^2 - y^2} = 9.682489722 \times 10^{-24} \rightarrow \sqrt{x(x + y) - y(x + y)} \]
\[ \approx 3.111669925 \times 10^{-12} \text{ [sec/cm]} \]

5.4 Deriving Electric Charge

Using the results of the previous section

\[ -\left[ \frac{\sqrt{1/(kt)}}{\sqrt{G}} \right] \left( \int \frac{1}{\alpha(\omega)} d\omega + i \int \frac{1}{\beta(\omega)} d\omega \right) = -\left[ \frac{\sqrt{1/(kt)}}{\sqrt{G}} \right] \int_0^a \sqrt{g} d\omega \]
\[ = \left[ \frac{\sqrt{sec/cm}}{\sqrt{G}} \right] -1 \left( \frac{1}{\sqrt{G}} \right)^{\omega-1} (\sqrt{G})^{-\omega} \left( \ln \left( \frac{1}{\sqrt{G}} \right) \right) + C \]
\[ = \left[ \frac{\sqrt{sec/cm}}{\sqrt{G}} \right] -1 \left( \frac{1}{\sqrt{G}} \right)^{\omega-1} (\sqrt{G})^{-\omega} \left( \ln \left( \frac{1}{\sqrt{G}} \right) \right) \bigg|_0^a \]
\[ = 3.111669925 \times 10^{-12} + i 1.526026276 \times 10^{-23} \]
\[ \rightarrow -\left[ \frac{\sqrt{1/(kt)}}{\sqrt{G}} \right] \left( \int \frac{1}{\alpha(\omega)} d\omega + i \int \frac{1}{\beta(\omega)} d\omega \right) \]
\[ \rightarrow \left( \sqrt{\frac{1}{v}}_D + i \sqrt{\frac{1}{v}}_L \right) \]
\[ = (-1.622335843 \times 10^{-19} - i 7.956265238 \times 10^{-31}) \left[ \frac{\sqrt{sec/cm}}{cm} \right], \]
\[ \sqrt{\frac{1}{v}}_D = \int \frac{1}{\alpha(\omega)} d\omega, \quad \sqrt{\frac{1}{v}}_L = \int \frac{1}{\alpha(\omega)} d\omega, \]

where \( a \approx 3.111669925 \times 10^{-12} \).

Note that the directly observable part of whatever this represents is extremely small. Projecting these numbers into instantiation space by letting \( \sqrt{x} = -1.622335843 \times 10^{-19} \) and \( \sqrt{y} = -7.956265238 \times 10^{-31} \), then
\[
\left(\frac{1}{\overline{v}}\right)_D + i \left(\frac{1}{\overline{v}}\right)_L = (\sqrt{x} + i \sqrt{y}) \sqrt{x + y}
\]

\[
\approx (-2.631973587 \times 10^{-38} - i 1.29077342710^{-49}) [sec/cm] \rightarrow \left(\frac{1}{\overline{v}}\right)_D
\]

\[
= (-2.631973587 \times 10^{-38})^2,
\]

\[
\left(\frac{i}{\overline{v}}\right)_L = -(1.29077342710^{-49})^2 \rightarrow e = \sqrt{\left(\frac{1}{\overline{v}}\right)_D^2 + \left(\frac{i}{\overline{v}}\right)_L^2}
\]

\[
= \sqrt{(-2.631973587 \times 10^{-38})^2 - (1.29077342710^{-49})^2}
\]

\[
= 2.631973587 \times 10^{-38} [sec/cm] \rightarrow \sqrt{e}
\]

\[
= \pm 1.622335843 \times 10^{-19} [\sqrt{sec/cm}]
\]

Evidently, \(\sqrt{e}\) is what physicists call 'electric charge'. Note that, like the speed of light, \(\sqrt{e}\) is a limit, not a constant. The number \(\sqrt{e}\) normally carries dimensions 'coulombs' i.e. \(1.622335843 \times 10^{-19} [C] = \sqrt{e}\). The experimentally measured value for electric charge is

\[
\approx 1.6021766208(98) \times 10^{-19} [C]
\]

Note that

\[
\frac{1.622335843 \times 10^{-19}}{3.111669925 \times 10^{-12}} \approx \sqrt{G} \approx 5.213714443 \times 10^{-8}
\]

The upshot is that

\[
- \frac{\sqrt{1/(kt)}}{\sqrt{G}} \left(\int \left(\frac{1}{\alpha(\omega)}\right)_D d\omega + i \int \left(\frac{1}{\beta(\omega)}\right)_L d\omega\right)
\]

\[
= -\frac{1}{\sqrt{G}} [\sqrt{sec/cm}] R_e \left(\frac{- \left(\frac{1}{\sqrt{G}}\right)^{\omega-1} (\sqrt{G})^{-\omega}}{\ln \left(-\frac{1}{\sqrt{G}}\right) - \ln \sqrt{G}}\right)_{0}^{3.111669925 \times 10^{-12}}
\]

\[
\rightarrow R_e \left(\int \left(\frac{1}{\alpha(\omega)}\right)_D d\omega + i \int \left(\frac{1}{\beta(\omega)}\right)_L d\omega\right) = \sqrt{e}
\]

\[
\approx -1.622335843 \times 10^{-19} [\sqrt{sec/cm}] \approx -a\sqrt{G},
\]

where \(a \approx 3.111669925 \times 10^{-12}\) and with the stipulation that the calculation ignores the imaginary part of the integration, which is comparatively small.
Hence,
\[ \frac{-1}{\sqrt{G}} R_e \left( \int_0^{3.111669925 \times 10^{-12}} \sqrt{g} \, d\omega \right) \approx 3.111669925 \times 10^{-12} \left[ \text{sec/cm} \right] \]

Let \( l = 1[cm/sec^2] \), then, multiplying both sides of the equation above by \( \sqrt{l} \),
\[ \frac{-1}{\sqrt{G}} R_e \left( \int_0^{3.111669925 \times 10^{-12}} \sqrt{g} \, d\omega \right) \approx 3.111669925 \times 10^{-12} \left[ \sqrt{\text{1/sec}} \right] \]

The calculations above suggest there is a missing acceleration factor somewhere. Moreover,
\[ R_e \left( \int_0^{3.111669925 \times 10^{-12}} \sqrt{g} \, d\omega \right) \approx -1.622335843 \times 10^{-19} \left[ \sqrt{\text{1/sec}} \right], \]

which suggests that electric charge is associated with a vibration.

### 5.5 The Other Side of the Equivalence Principle

The law of the equivalence of gravitational with inertial mass can be written in the form
\[ a = \frac{GM}{r^2} \]

In image theory, this would be written
\[ a_D + ia_L = \frac{(GM)}{r^2}_D + i \left( \frac{(GM')}{r'^2} \right)_L, \]

where \( (GM/r^2)_D \) represents the dark- and \( (GM'/r'^2)_L \) the light-part of the acceleration.

If \( a_D \rightarrow \sqrt{a_D} \) and \( a_L \rightarrow \sqrt{a_L} \), then
\[ \sqrt{a_D} + i \sqrt{a_L} = \sqrt{\frac{(GM)}{r^2}}_D + i \sqrt{\frac{(GM')}{r'^2}}_L \]

Let
\[ \sqrt{\frac{(GM)}{r^2}}_D + i \sqrt{\frac{(GM')}{r'^2}}_L = \sqrt{k} \sqrt{f} = \sqrt{k} \left( -\frac{1}{\sqrt{G}} \right)^{1-\omega} \left( \sqrt{G} \right)^{\omega}, \quad \omega = vt, \]

where the expression on the right-side is the fundamental function of image theory. Note that the acceleration is a function of time, \( v = 1 \) and \( k = 1[cm/sec^2] \).
Multiplying through by $1/\sqrt{G}$, the equation above becomes

$$[\sqrt{l}] \left( \sqrt{\left( \frac{M}{r^2} \right)_D} + i \sqrt{\left( \frac{M'}{r'^2} \right)_L} \right) = [\sqrt{k}] \left( -\frac{1}{\sqrt{G}} \right)^{1-\omega} \left( \sqrt{G} \right)^{\omega-1},$$

where $l = 1[cm^3/gm sec^2]$. The units \( \langle l \rangle \) are introduced because $G$ is dimensionless.

Now

$$\sqrt{\left( \frac{M}{r^2} \right)_D} + i \sqrt{\left( \frac{M'}{r'^2} \right)_L} \rightarrow \left( \frac{M}{r^2} \right)_D = a(\omega)_D, \quad \left( \frac{M'}{r'^2} \right)_L = a(\omega)_L$$

represents an acceleration, which is a function of $\omega$. If the above equation is integrated with respect to $\omega$, then

$$\sqrt{k} \int a(\omega)_D d\omega + i \sqrt{k} \int a(\omega)_L d\omega = \sqrt{v_D} + i \sqrt{v_L} = \sqrt{k} \int \left( -\frac{1}{\sqrt{G}} \right)^{1-\omega} G^{\omega-1} d\omega$$

$$= \left[ \frac{\sqrt{k}}{\sqrt{v}} \right] \int \left( -\frac{1}{\sqrt{G}} \right)^{1-\omega} G^{\omega-1} d\omega = \left[ \sqrt{cm/sec} \right] \left[ \frac{\left( -\frac{1}{\sqrt{G}} \right)^{1-\omega} G^{\omega-1}}{-\ln \left( -\frac{1}{\sqrt{G}} \right) + \ln \sqrt{G}} + C, \right.$$

$$v = 1$$

From a dimensional standpoint, the integration represents some kind of gravitational velocity, where \( v_D \) and \( v_L \) represent the dark- and light-parts respectfully.

Let $\omega$ vary from 0 to $\infty$, then by direct calculation

$$\sqrt{v_D} + i \sqrt{v_L} = \left[ \sqrt{k \times sec} \right] \left[ \frac{\left( -\frac{1}{\sqrt{G}} \right)^{1-\omega} G^{\omega-1}}{-\ln \left( -\frac{1}{\sqrt{G}} \right) + \ln \sqrt{G}} \right]_{0}^{\infty}$$

$$\approx \left( -1.087337476 \times 10^{13} + i1.018514029 \times 10^{12} \right) \left[ \sqrt{cm/sec} \right]$$

Letting $\sqrt{x} = -1.087337476 \times 10^{13}$ and $\sqrt{y} = 1.018514029 \times 10^{12}$ and projecting these numbers into instantiation space gives

$$\left( \sqrt{x} + i \sqrt{y} \right) \sqrt{x + y} = \tilde{v}_D + i \tilde{v}_L$$

$$= \left( -1.187478315 \times 10^{26} + i1.112316415 \times 10^{25} \right) [cm/sec]$$

Now

$$\tilde{v}_D^2 = (-1.187478313 \times 10^{26})^2, \quad -\tilde{v}_L^2 = -(1.112316413 \times 10^{25})^2$$
Hence,

$$\sqrt{\tilde{v}_D^2 - \tilde{v}_L^2} = 1.182257276 \times 10^{26} \text{ [cm/sec]}$$

In physics, $v^2$ is normally proportional to the energy of a system. So, the equation above is, evidently, the square-root of the difference between dark- and light-gravitational energy. Interestingly,

$$\frac{1}{\sqrt{\tilde{v}_D^2 - \tilde{v}_L^2}} = 8.458395819 \times 10^{-27} \text{ [sec/cm]}$$

Note that the numerical value of this number is close to Planck’s constant, which measures $\approx 6.62 \times 10^{-27} \text{ [gm cm}^2/\text{sec]}. \text{ Keep in mind that } \tilde{v}_D, \tilde{v}_L \text{ are limiting velocities and not constants.}$

### 5.5.1 Planck’s Constant in Instantiation Space

Similar to the previous sections, the inverse of the fundamental equation will be employed. So,

$$\sqrt{\frac{r^2}{(l)GM}}_D + i \sqrt{\frac{r'^2}{(l)GM'}}_L = \sqrt{\frac{1}{k}} \int \frac{1}{\sqrt{\frac{1}{\beta(\omega)}}} d\omega = \sqrt{\frac{1}{\nu}} + i \sqrt{\frac{1}{\nu}}$$

$$\omega = v t$$

If the equation above is integrated with respect to $\omega$:

$$\left[ \sqrt{\frac{1}{kt}} \right] \left( \int \frac{1}{\alpha(\omega)} \right)_D d\omega + i \int \frac{1}{\beta(\omega)} L d\omega = \sqrt{\frac{1}{\nu}} D + i \sqrt{\frac{1}{\nu}} L$$

$$= \left[ \sqrt{\text{sec/cm}} \right] \left( - \frac{1}{\sqrt{G}} \right)^{\omega-1} (\sqrt{G})^{1-\omega} \ln \left( -\frac{1}{\sqrt{G}} \right) - \ln \sqrt{G} + C, \quad t = 1, \quad \omega = vt$$
Continuing,

\[
\left[ \sqrt{\text{sec/cm}} \right] \frac{\left( -\frac{1}{\sqrt{G}} \right) \omega^{-1} (\sqrt{G})^{1-\omega}}{\ln \left( \frac{-1}{\sqrt{G}} \right) - \ln \sqrt{G}} \right|_0^{3.111669924 \times 10^{-12}}
\]
\[
\approx (-8.458395814 \times 10^{-27} - i4.148169497 \times 10^{-38}) \left[ \sqrt{\text{sec/cm}} \right]
\]

Projecting these numbers into instantiation space by letting

\[
\sqrt{x} = -8.458395814 \times 10^{-27}, \quad \sqrt{y} = -4.148169497 \times 10^{-38},
\]

then

\[
\left( \frac{1}{\sqrt{\nu}} \right) _D + i \left( \frac{1}{\sqrt{\nu}} \right) _L = \sqrt{x} + i\sqrt{y} \sqrt{x + y}
\]
\[
\approx (-7.154445975 \times 10^{-53} - i3.508685951 \times 10^{-64}) \left[ \text{sec/cm} \right]
\]

Hence,

\[
\left( \frac{1}{\sqrt{\nu}} \right) _D^2 = (-7.154445975 \times 10^{-53})^2,
\]
\[
\left( \frac{1}{\sqrt{\nu}} \right) _L^2 = (-i3.508685951 \times 10^{-64})^2 \rightarrow \hbar = \sqrt{\left( \frac{1}{\sqrt{\nu}} \right) _D^2 + \left( \frac{1}{\sqrt{\nu}} \right) _L^2}
\]
\[
= \sqrt{(-7.154445973 \times 10^{-53})^2 - (3.508685950 \times 10^{-64})^2}
\]
\[
= 7.154445975 \times 10^{-53} \left[ \text{sec/cm} \right] \rightarrow \sqrt{\hbar}
\]
\[
\approx \pm 8.458395814 \times 10^{-27} \left[ \sqrt{\text{sec/cm}} \right]
\]

Evidently, \( \sqrt{\hbar} \) is what physicists call ‘Planck’s constant’, although, in this case, it is a limiting value. The number \( \sqrt{\hbar} \) normally carries units of \( [gm - cm^2/sec] \) and its numerical value is \( \approx 6.62 \times 10^{-27} \).

Again, note that

\[
\frac{8.458395814 \times 10^{-27}}{1.622335843 \times 10^{-19}} \approx \sqrt{G} = 5.213714442 \times 10^{-8}
\]
Hence,

\[-\frac{1}{\sqrt{G}}\left(\left[\sqrt{\frac{1}{kt}}\right]\left(\int \frac{1}{a(\omega)} d\omega + i \int \frac{1}{a(\omega)} d\omega\right)\right) = \frac{-1}{\sqrt{G}}\left(\int \frac{1}{v_D} d\omega + i \int \frac{1}{v_L} d\omega\right)\]

\[\rightarrow R_e \left(\left(-\frac{1}{\sqrt{G}}\right)^{\omega-1} \left(\sqrt{G}\right)^{1-\omega} \int_0^{3.111\times10^{-12}} \left(-\frac{1}{G}\right)^{\omega-1} (G)^{-\omega+1} d\omega\right)\]

\[\approx -\sqrt{G} (1.622335843 \times 10^{-19}) \sqrt{sec/cm} = -\sqrt{Ge}\]

The calculation ignores the imaginary part of the integration as too small to count. Hence,

\[-\frac{1}{\sqrt{G}} R_e \left(\int_0^{3.111\times10^{-12}} \sqrt{(-\frac{1}{G})^{\omega-1} (G)^{-\omega+1}} d\omega\right) \approx 1.622335843 \times 10^{-19} \sqrt{sec/cm}\]

Let \(l = 1[cm/sec^2]\), then multiplying both sides of the equation above by \(\sqrt{l}\),

\[-\frac{1}{\sqrt{G}} R_e \left(\int_0^{3.111\times10^{-12}} \sqrt{(-\frac{1}{G})^{\omega-1} (G)^{-\omega+1}} d\omega\right) \approx 1.622335843 \times 10^{-19} \sqrt{1/sec}\]

Again, the calculations above suggest there is a missing acceleration factor. Moreover,

\[\int_0^{3.111\times10^{-12}} \sqrt{(-\frac{1}{G})^{\omega-1} (G)^{-\omega+1}} d\omega \approx -8.458395815 \times 10^{-27} \sqrt{1/sec}\]

which suggests that Planck’s constant is associated with a vibration.

### 5.6 Conjecture on Mass

In classical physics, ‘mass’ is a physical property of a body, but it is also associated with a body’s resistance to being accelerated by an external force. It likewise determines the strength of the mutual gravitational attraction between two bodies. In Newtonian physics, intuitively, mass is the amount of matter a body possesses. A massive body at rest has an amount of energy proportional to the speed of light and all forms of energy are subject to gravitational effects.

Modern thinking on the subject no longer regards the concept of ‘mass’ as fundamental, because its definition is elusive. Repeated experiments since the 17th century have
demonstrated that inertial and gravitational mass are equal, a law embodied in the equivalence principle of general relativity.

At the quantum level, mass is conceived as a disturbance in the matter fields that permeate space and time. Particles gain mass through the Higgs mechanism, where mass is associated with ‘symmetry breaking’. For example, weakly interacting particles are governed by an $SU(2)$ symmetry. When this symmetry is broken, the gauge bosons associated with this interaction gain mass.

Recall that the fundamental function was used to describe an acceleration i.e.

$$\sqrt{\left(\frac{d^2S}{dt^2}\right)_D} + i \sqrt{\left(\frac{d^2S}{dt^2}\right)_L} = \sqrt{k} \left(-\frac{1}{\sqrt{G}}\right)^{1-\omega} (\sqrt{G})^\omega, \quad k = [cm/sec^2]$$

To make a sensible physics equation, a number ‘ν’ was introduced having dimensions ‘[1/sec]’, making the product ‘$\omega = \nu t$’ dimensionless, suggesting that $\nu$ represents a vibration ([cycles per sec]). To measure time in the simplest manner, $\nu$ was set equal to 1 so that 1 [cycle] equals 1 [sec].

If $\nu > 1$, then, the limit speeds of both the real and imaginary parts of the calculation are numerically less than if $\nu = 1$. For instance, if $\nu = 5$, then

$$\left(\frac{dS}{dt}\right)_D + i \left(\frac{dS}{dt}\right)_L \approx (-1.291160289 \times 10^{10} + i1.209435798 \times 10^9) [cm/sec],$$

as opposed to

$$\left(\frac{dS}{dt}\right)_D + i \left(\frac{dS}{dt}\right)_L \approx (-3.227900723 \times 10^{11} + i3.023589495 \times 10^{10}) [cm/sec],$$

when $\nu = 1$. This suggests that the speed of light depends on whether or not $\nu$ is in sync with time.

If $\nu = 1700$, then

$$\left(\frac{dS}{dt}\right)_D + i \left(\frac{dS}{dt}\right)_L \approx (-1.116920664 \times 10^5 + i1.046224737 \times 10^4) [cm/sec]$$

In general, the greater the vibration (ν), the slower the maximum dark- and observational limit speeds. Whatever these objects (objects with $\nu > 1$) represent, their maximum speed is slower than light speed, suggesting that these kinds of objects, if they exist, are massive.

Evidently, $\nu \ll 1$, since such objects, if they existed, would exceed the limiting speeds of both dark- and light-light in contradiction to the principle that nothing travels faster than light.
Moreover, conducting this same exercise on the small-scale does not have an appreciable effect on the outcome. For instance, if \( \nu = 5 \), then
\[
\sqrt{\left(\frac{dt}{d\bar{S}}\right)^2_D + \left(i \frac{dt}{d\bar{S}}\right)^2_L} \to \sqrt{e} \approx 1.622335843 \times 10^{-19} \left[\sqrt{1/\text{sec}}\right],
\]
which is virtually identical to the result when \( \nu = 1 \). Moreover, if \( \nu = 1700 \), then
\[
\sqrt{\left(\frac{dt}{d\bar{S}}\right)^2_D + \left(i \frac{dt}{d\bar{S}}\right)^2_L} \to \sqrt{e} \approx 1.622335843 \times 10^{-19} \left[\sqrt{1/\text{sec}}\right],
\]
suggesting that noticeable effects on the small scale occur only if \( \nu \) is extremely large.

Moreover, it appears that
\[
\lim_{\nu \to \infty} \left[ \left(\frac{d\bar{S}}{dt}\right)^2_D + i \left(\frac{d\bar{S}}{dt}\right)^2_L \right] = 0
\]
In other words, as \( \nu \to \infty \), the object becomes immovable.

5.7 Concluding Remarks

Objective mathematics represents an abstract physical space containing an observer and an independently existing external world relative to that observer. The concept of 'interference', in general, implies that human beings have an unfaithful knowledge of the external world. A direct empirical observation of an external object, not part of the human body, depends on the extent to which the external object has something in common with the observer's body. The perceptible part of an external object is called the 'light-part'. The imperceptible part is called 'dark'.

The form \( \langle \sqrt{a}, \sqrt{b} \rangle \sqrt{a + b} \) embodies the information an observer possesses of an external object after an observation. The quantity \( \sqrt{a + b} \) represents an interference factor, which signifies the extent to which an observer is denied faithful information about an external object.

The function
\[
f(t) = \left( -\frac{1}{G} \right)^{1-t} G^t,
\]
where \( G \) is a real constant, is called the 'fundamental' function of image theory. All physical processes in image theory are derivable from this function. If \( G \to \sqrt{G} \), \( \sqrt{G} \) is approximately numerically equal to Newton's gravitational constant. Integration of the fundamental function, with a subsequent projection into instantiation space, produces the limiting speed of light. But such a calculation comes with an extra ingredient – the limiting speed of 'dark-light', light that is not detectable.
The fundamental function ‘$f$’ is associated with the physics on the large-scale. It’s exponential inverse ‘$g$’ involves physics on the small-scale. However,

\[ \int_{0}^{\infty} g \, d\omega = \infty \]

Avoiding the divergence requires renormalization, which necessitates specifying a cutoff value. The cutoff is hypothesized as $\approx 3.111669925 \times 10^{-12}$. Applying the cutoff along with a projection into instantiation space, produces what appears to be a limiting ‘electric charge’.

By applying the law of the equivalence of gravitational with inertial mass

\[ a = \frac{GM}{r^2}, \]

then integrating the fundamental function with subsequent projection into instantiation space, produces a kind of limiting gravitational velocity. The numerical difference between the dark- and light-parts of this velocity $\sqrt{v_D^2 - v_L^2}$ is enormous

\[ \approx 1.182257278 \times 10^{26} \text{ [cm/sec]} \]

Of note,

\[ \frac{1}{\sqrt{v_D^2 - v_L^2}} \approx 8.458395811 \times 10^{-27}, \]

which is close to the value of Planck’s constant ($\approx 6.62 \times 10^{-27}$ [gm cm$^2$/sec]).

Remarkably, using the inverse of the fundamental function and the cutoff, integration and projection into instantiation space produces the value $\approx 8.458395811 \times 10^{-27}$. Hence, Planck’s constant, or what is more appropriately called ‘Planck’s limit’, appears to be the inverse of a gravitational velocity.

Moreover, Planck’s limit and electric charge only differ by a constant, namely $\sqrt{G}$, the gravitational constant. In order to make the dimensions consistent in the calculations, $G$ must be associated with a factor of acceleration. This suggests a force is missing from the analysis.
**Chapter 6**

*Image Theory and Special Relativity*

“One of the most exciting things about dark energy is that it seems to live at the very nexus of two of our most successful theories of physics: quantum mechanics, which explains the physics of the small, and Einstein's Theory of General Relativity, which explains the physics of the large, including gravity.”

- Adam Riess

6.0 Introduction

The theory of relativity emerges from image theory through 'instantiation'. Recall that a complex pair ‘〈x, y〉’ requires an instantiation i.e.

\[ \langle x, y \rangle \xrightarrow{f} \langle \sqrt{x} \sqrt{x+y}, \sqrt{y} \sqrt{x+y} \rangle \]

If \( f \) is a valid instantiation, then so is

\[ \langle x, y \rangle \xrightarrow{-f} \langle \sqrt{y} \sqrt{x+y}, -\sqrt{x} \sqrt{x+y} \rangle \]

obtained by multiplying \( \langle \sqrt{x} \sqrt{x+y}, \sqrt{y} \sqrt{x+y} \rangle \) by \(-i\). In other words, valid instantiations may not be unique, but are equivalent. This chapter discusses the relationship between image theory and the theory of special relativity.

6.1 The Principle of Relativity as an Instantiation

In the previous chapter, the following relationships were deduced:

\[ i \left( \frac{d\overline{S}}{dt} \right)_L = ic, \quad \left( \frac{d\overline{S}}{dt} \right)_D = -c' \rightarrow \left( \left( \frac{d\overline{S}}{dt} \right)_D, \left( \frac{d\overline{S}}{dt} \right)_L \right) = (-c', c), \]

where \( (d\overline{S}/dt)_L \) and \( (d\overline{S}/dt)_D \) represent the instantiations of the speeds of light- and dark-light respectively. Let

\[ -i \left( \left( \frac{d\overline{S}}{dt} \right)_D, \left( \frac{d\overline{S}}{dt} \right)_L \right) = \left( \left( \frac{d\overline{S}'}{dt'} \right)_D, -\left( \frac{d\overline{S}'}{dt'} \right)_L \right) = (c, c') \]

(see fig. 6.1-1 a), b)). Therefore,

\[ -\left( \frac{d\overline{S}}{dt} \right)_L^2 = -c^2, \quad \left( \frac{d\overline{S}'}{dt'} \right)_L^2 = c'^2 \rightarrow -d\overline{S}_L^2 = -c^2 \, dt^2, \quad (d\overline{S}_L')^2 = (c \, dt')^2 \]
Adding the last two equations on the right above leaves
\[ (dS_L')^2 - dS_L^2 = (c\, dt')^2 - c^2\, dt^2 \rightarrow dS_L^2 - c^2\, dt^2 = (dS_L')^2 - (c\, dt')^2 \]

The last equation on the right above can be expanded to three spatial dimensions by letting
\[ dS_L^2 = dx_L^2 + dy_L^2 + dz_L^2, \quad (dS_L')^2 = (dx_L')^2 + (dy_L')^2 + (dz_L')^2 \]

The result is
\[ dx_L^2 + dy_L^2 + dz_L^2 - c^2\, dt^2 = (dx_L')^2 + (dy_L')^2 + (dz_L')^2 - (c\, dt')^2, \]

which is the infinitesimal version of the 'principle of relativity' derived through instantiation. All the outcomes of special and general relativity originate from this principle (see Book III, Chapter 9). When only considering inertial systems, the \( d's \) can be eliminated and the equation above can be written
\[ x_L^2 + y_L^2 + z_L^2 - c^2\, t^2 = (x_L')^2 + (y_L')^2 + (z_L')^2 - (ct')^2 \]

According to the principle of relativity, two observers moving with a constant speed relative to one another will not agree on the location or time of an event in space-time, but will agree on the 'space-time interval', represented by the equation above.

According to the principle of relativity, two observers moving with a constant speed relative to one another will not agree on the location or time of an event in space-time, but will agree on the 'space-time interval', represented by the equation above.

![Figure 6.1-1](image)

In the context of image theory, simply stated, the 'principle of relativity' is the acknowledgement that one valid instantiation cannot be favored over another.

There is a dark 'relativity principle' identical to the one for light-light i.e.
\[ \frac{dS_D^2}{dt^2} = (c')^2, \quad -\frac{(dS_D')^2}{(dt')^2} = -(c')^2 \rightarrow dS_D^2 = (c')^2\, dt^2, \quad -(dS_D')^2 = -(c'\, dt')^2 \]

Repeating the same process as was done for light-light results in
\[ dx_D^2 + dy_D^2 + dz_D^2 - c^2\, dt^2 = (dx_D')^2 + (dy_D')^2 + (dz_D')^2 - (c'\, dt')^2 \]
6.1.1 The Principle of Relativity Interpreted

The principle of relativity requires that all physical laws be independent of the disposition of an observer. An observer at rest in an inertial reference frame cannot determine an absolute speed or direction of travel in space, but only speed or direction relative to some other inertial frame. In other words, if two observers are moving with a constant velocity relative to one another, both observers have equal right to claim the “at rest” disposition, while declaring that the other observer is the one moving. In essence, the principle of relativity forbids a preferred perspective.

6.1.2 The Principle of Relativity for Combined Dark- and Light-Light

Finding a ‘principle of relativity’ for combined light- and dark-light is fraught with difficulties. If the motion is confined to the $x$-direction, the principles of relativity for light- and dark-light are written

$$dx_L^2 - c^2 dt^2 = (dx'_L)^2 - (c dt')^2, \quad dx_D^2 - (c' dt)^2 = (dx'_D)^2 - (c'dt')^2$$

respectfully. It is always possible to choose coordinates such that $x'_D = 0$ and $x'_L = 0$ leaving

$$dx_L^2 - c^2 dt^2 = -(c dt')^2, \quad dx_D^2 - (c' dt)^2 = -(c' dt')^2$$

Subtracting the first equation from the second gives

$$dx_D^2 - dx_L^2 - [(c')^2 - c^2] dt^2 = -[(c')^2 - c^2] dt'^2,$$

which can be written

$$-(dx_D^2 - dx_L^2) + (c'^2 - c^2) dt^2 = (c'^2 - c^2) (dt')^2 \rightarrow \left(1 - \frac{v_D^2 - v_L^2}{c'^2 - c^2}\right) dt^2 = (dt')^2$$

$$\rightarrow \sqrt{1 - \frac{v_D^2 - v_L^2}{c'^2 - c^2}} dt = dt', \quad v_D^2 = \frac{dx_D^2}{dt^2}, \quad v_L^2 = \frac{dx_L^2}{dt^2},$$

where $v_D$ represents the velocity relative to dark-light and $v_L$ represents the velocity relative to light-light. If $v_D, v_L$ are assumed arbitrary values, then according to the rules of relativity, the time on a clock in constant motion, measured by a clock at rest, runs slow by a factor

$$\sqrt{1 - \frac{v_D^2 - v_L^2}{c'^2 - c^2}}$$
If \( v_D < v_L \), then
\[
\sqrt{1 - \frac{v_D^2}{c^2}} < 1 - \frac{v_L^2}{c^2},
\]
in which case the “at rest” clock would measure the clock in motion as running slower, directly contradicting of the theory of relativity.

**6.1.2.1 The Contradiction in Combining Dark- with Light-Light**

If a light-light beam is sent from \( A \) at time \( t = t' = 0' \), it will reach point ‘\( B' \) at time \( t' = \frac{AB}{c} \), measured by a stationary clock labeled ‘\( t'' \). But a dark-light beam would reach point ‘\( B'' \) in time \( t' = \frac{AB'}{c} \) (see fig. 6.1.2.1-1). An observer moving with constant velocity ‘\( v_L \)', at time ‘\( t = t' = 0' \), travels from ‘\( B' \) to ‘\( x' \) in time ‘\( t \)', measured on the moving clock, labeled ‘\( t \).’ Consider another observer stationed at ‘\( B' \), who, at time ‘\( t = t' = 0' \), moves with constant velocity ‘\( v_D \)' from ‘\( B' \)' to ‘\( y' \) in time ‘\( t' \)', measured on the moving clock. Assuming \( v_D = v_L \), then
\[
\frac{AB}{c} = \frac{AB'}{c'} = t' \rightarrow \frac{AB'}{c'} = \frac{AB'}{k} = ct', \quad c' = kc
\]
Therefore,
\[
(AB)^2 + (Bx)^2 = (Ax)^2 \rightarrow (AB)^2 = (ct)^2 - (v_L t)^2 \rightarrow \left(\frac{AB}{c}\right)^2 = \left(1 - \frac{v_L^2}{c^2}\right) t^2 \rightarrow \frac{t'}{\sqrt{1 - \frac{v_L^2}{c^2}}} = t,
\]
\[
\frac{AB}{c} = t'
\]

![Figure 6.1.2.1-1](image-url)
If the dark-light moves along the path \( \overline{AY} \), then, according to the theory of relativity,

\[
(AB')^2 + (B'y)^2 = (Ay)^2 
\]

\[
= (AB')^2 + (v_D t)^2 = (c't)^2 \rightarrow (AB')^2 = (c'' - v_D^2)t^2 \rightarrow \left( \frac{AB'}{c'} \right)^2
\]

\[
= \left(1 - \frac{v_D^2}{c'^2}\right) t^2 \rightarrow \frac{t'}{\sqrt{1 - \frac{v_D^2}{c'^2}}} = t, \quad \overline{AB'} = t'
\]

But then

\[
\left(1 - \frac{v_D^2}{(c')^2}\right) t^2 = \left(1 - \frac{v_L^2}{c^2}\right) t^2 \rightarrow 1 - \frac{v_D^2}{(c')^2} = 1 - \frac{v_L^2}{c^2} \rightarrow \frac{v_D^2}{(c')^2} = \frac{v_L^2}{c^2} \rightarrow v_D = k v_L, \quad c' = kc
\]

Hence, \( v_D \neq v_L \), directly contradicting the assumption \( 'v_D = v_L' \).

Suppose \( v_L = c \), then, according to the postulates of relativity, no time should tick off a clock moving with the speed of light-light as measured by a stationary clock. If \( v_D = c' > c \), then, intuitively, the clock moving with dark-light speed should run slower than a clock moving with light-light speed. Intuitively, then, a clock moving with dark-light speed would run backwards in time or, at least, produce an imaginary time as measured by a stationary clock.

**6.1.3 Invariance of Dark-Light Relative to Light-Light**

The problem is hypothetical, since dark-light is not directly detectable. But theoretically, the contradiction can be removed by the following line of argument. Given a distance \( 'L' \), on a stationary clock labeled \( 't'' \), when \( t' = 0 \), let a light-light beam move from left to right toward \( E_1 \) while a dark-light beam moves from left to right toward \( E_2 \) (see fig. 6.1.3-1). Likewise, when \( t' = 0 \), an object moves from right to left toward \( E_1 \) with constant velocity \( 'v' \), relative to the stationary clock,

![Figure 6.1.3-1](image)

while a second object moves from right to left toward \( E_2 \) with velocity

\[
\frac{v}{k'} \quad k = \frac{c'}{c}
\]

124
The event ‘$E_1$’ occurs when the light-light beam reaches the first object, while event ‘$E_2$’ occurs when the dark-light beam reaches the second object. In order for the two events ‘$E_1$’ and ‘$E_2$’ to be simultaneous, as measured by the stationary clock ‘$t'$’, the velocity of the object measured relative to dark-light must be $v/k$. To see this, note that

$$L - ct' = vt', \quad L - c't' = \frac{v}{k}t' \rightarrow \frac{L - ct'}{v} = \frac{k(L - c't')}{v} \rightarrow L - ct' = k(L - c't') \rightarrow t'$$

Now

$$L = \frac{v}{k}t' + c't' \rightarrow c(k + 1)t' = \left(\frac{v}{k} + c'\right)t' \rightarrow c(k + 1) = \left(\frac{v}{k} + kc\right) \rightarrow \frac{v}{k} = c \rightarrow v = c$$

Likewise,

$$L = vt' + c't' = (v + c)t' \rightarrow c(k + 1)t' = (v + c)t' \rightarrow kc + c = v + c \rightarrow v = kc = c$$

Thus, if events ‘$E_1$’ and ‘$E_2$’ happen simultaneously as measured by a clock at rest, then the events do not occur at the same spatial location. However, the spatial difference ‘$D_{E_1} - D_{E_2}$’ between the events, one measured relative to dark-light, the other measured relative to light-light, is a constant. To see this,

$$D_{E_1} - D_{E_2} = vt' - \frac{v}{k}t' = \left(\frac{k - 1}{k}\right)vt' \rightarrow \frac{D_{E_1} - D_{E_2}}{vt'} = \frac{k - 1}{k}$$

### 6.2 Deriving the Lorentz Transformations for Dark-Light

Suppose there are three observers labeled ‘$P_1, P_2, P_3$’ at rest relative to one another in a coordinate system ‘$S''$’, spaced 1 unit apart (see fig. 6.2-1). Suppose $P_1$ and $P_3$ agree to send a light signal to $P_2$ when their clocks read a certain time. If $P_1$ sends a dark-light signal, it will reach $P_2$ in time ‘$t'$’ at point ‘$a'$’. In order for $P_3$ to send a dark-light signal that will reach the point ‘$a''$’ at time ‘$a''$’, in other words, will reach $P_2$ simultaneously with the light signal sent by $P_1$, $P_3$’s light ray must make the same angle with the $x$-axis as if it was sent from ‘$O$’ (see fig. 6.2-1).
Note that the speed of dark-light is \( c' = kc \), where \( k \approx 10.67 \). The equation of the line running through 'Oa' is \( x = t \). The equation of the line running through 'Oa'' is \( x = kt \), since \( kc = c' \), \( c = 1 \). Hence,

\[
kt - vt = 1 \to t = \frac{1}{k - v}
\]

Since \( x = kt \) at \( a' \), then

\[
t = \frac{1}{k - v} \to kt = \frac{k}{k - v}
\]

If the slope of the line running through \( Oa' \) is \( k \), the equation of the line running through \( a'b' \) is \( x = -kt + C \), \( C = a \text{ constant} \) or

\[
x + kt = C \to \frac{k}{k - v} + \frac{k}{k - v} = \frac{2k}{k - v} = C
\]

At \( b' \),

\[
x - vt = 2, \quad x + kt = \frac{2k}{k - v}
\]

Solving the last two equations simultaneously,

\[
vt + kt = \frac{2k}{k - v} - 2 \to (v + k)t = \frac{2k}{k - v} - \frac{2(k - v)}{k - v} = \frac{2v}{k - v} \to t = \frac{2v}{k^2 - v^2}
\]

Moreover,

\[
x + kt = \frac{2k}{k - v} \to x = \frac{2k}{k - v} - \frac{2kv}{k^2 - v^2} = \frac{2k(k + v)}{k^2 - v^2} - \frac{2kv}{k^2 - v^2} = \frac{2k^2}{k^2 - v^2}
\]
Therefore,

\[
\frac{t}{x} = \frac{2v}{k^2 - v^2} = \frac{v}{k} \Rightarrow kt = \frac{vx}{k}
\]

The time ‘\(kt\)’ is when the “at rest” observers in \(S'\) will agree that their clocks are synchronous.

Note that the line ‘\(x = vt/k\)’ has an inverse slope relative to ‘\(t = vx/k\)’ i.e.

\[
t = \frac{kx}{v} \Rightarrow x = \frac{vt}{k}
\]

So evidently,

\[
t - \frac{vx}{k} = t', \quad x - \frac{vt}{k} = x',
\]

since it is always possible to choose coordinates such that \(t' = 0\) and \(x' = 0\).

Remembering that if \(x = kt\), then \(x' = kt'\), since the speed of light must be the same to all observers. Therefore,

\[
x' = \left(x - \frac{vt}{k}\right)f(v), \quad t' = \left(t - \frac{vx}{k}\right)g(v),
\]

where \(f(v)\) and \(g(v)\) are functions of \(v\) that would make \(x' = kt'\). But the above equations show that for \(x'\) to be equal to \(kt'\), \(f(v) = g(v)\). Solving

\[
x' = \left(x - \frac{vt}{k}\right)f(v), \quad t' = \left(t - \frac{vx}{k}\right)f(v)
\]

simultaneously for \(x\) and \(t\) respectfully leaves

\[
x = \left(x' + \frac{vt'}{k}\right)f(v) \quad t = \left(t' + \frac{vx'}{k}\right)f(v)
\]

The equations ‘\(x'\), \(t'\), \(t\), \(x\)’ can be made compatible if plugging the equation for \(x'\) into the equation for \(x\) gives \(x = x\). The same can be said of \(t\) and \(t'\). Hence,

\[
x = \left(x' + \frac{vt'}{k}\right)f = \left(\left(x - \frac{vt}{k}\right)f + \frac{v}{k}\left(t - \frac{vx}{k}\right)f\right) = \left(x - \frac{vt}{k} + \frac{vt}{k} - \frac{v^2x}{k^2}\right)f^2 \Rightarrow x
\]

\[
= \left(x - \frac{v^2x}{k^2}\right)f^2 \Rightarrow f = \frac{1}{\sqrt{1 - \frac{v^2}{k^2}}}
\]
6.2.1 The Lorentz Transformations for Dark-Light

The Lorentz transformations for dark-light are derived in the same manner as for light-light. First, if the equations are compliant with the postulates of relativity, and if the motion is in the $x$-direction only, then

$$x' = f \left( x - \frac{vt}{k} \right), \quad x = f \left( x' + \frac{vt'}{k} \right), \quad y' = y, \quad z' = z$$

where $f$ does not depend on $x$ or $t$, but may depend on $v$. Keeping in mind that $x, t \in S$ and $x', t' \in S'$, where the systems 'S' and 'S'' are considered two coordinate systems one in constant linear motion relative to the other, then

$$x = f \left( f \left( x - \frac{vt}{k} \right) + \frac{vt'}{k} \right) = f^2 \left( x - \frac{vt}{k} \right) + \frac{vt'}{k} \to t' = \left( \frac{1 - f^2}{vf} \right) kx + ft$$

The last equation follows after a series of algebraic manipulations. Recall that the postulates of relativity demand that $x = ct$, $x' = ct'$. Hence,

$$f \left( x - \frac{vt}{k} \right) = c \left[ \left( \frac{1 - f^2}{vf} \right) kx + ft \right] = \left( \frac{1 - f^2}{vf} \right) kcx + fct$$

Solving for $x$ leads to

$$\left[ f - \left( \frac{1 - f^2}{f} \right) \frac{kcx}{v} \right] x = fct + \frac{fvt}{k} \to x = \frac{fct + \frac{fvt}{k}}{f - \left( \frac{1 - f^2}{f} \right) \frac{kcx}{v}} = \frac{ct + \frac{vct}{kc}}{1 - \left( \frac{1}{f^2} - 1 \right) \frac{kcx}{v}}$$

$$= \left[ \frac{1 + \frac{v}{kc}}{1 - \left( \frac{1}{f^2} - 1 \right) \frac{kcx}{v}} \right] ct$$

In order for $x = ct$ to be true, the quantity in square brackets must equal 1. Solving,

$$\left( \frac{1 + \frac{v}{kc}}{1 - \left( \frac{1}{f^2} - 1 \right) \frac{kcx}{v}} \right) = 1 \to \frac{v}{kc} = 1 - \left( \frac{1}{f^2} - 1 \right) \frac{kcx}{v} \to \frac{1}{f^2} = 1 - \frac{v^2}{k^2c^2} \to f = \frac{1}{\sqrt{1 - \frac{v^2}{k^2c^2}}}$$

The following sets of equations are the Lorentz transformations for light- and dark-light, the four dark transformations appearing on the right-half below:

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad x = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}} \quad x' = \frac{x - \frac{v}{k}t}{\sqrt{1 - \frac{v^2}{k^2c^2}}}, \quad x = \frac{x' + \frac{v}{k}t'}{\sqrt{1 - \frac{v^2}{k^2c^2}}}$$
\[ t' = t - \frac{v}{c^2} x \sqrt{1 - \frac{v^2}{c^2}} \]
\[ t = t' + \frac{v}{c^2} x' \sqrt{1 - \frac{v^2}{c^2}} \]
\[ t' = t - \frac{v}{c^2} x \sqrt{1 - \frac{v^2}{c^2}} \]
\[ t = t' + \frac{v}{c^2} x' \sqrt{1 - \frac{v^2}{c^2}} \]

6.2.2 Dark- and Light-Light Compatibility

Evidently, in the case of dark-light,
\[ \lim_{v \to k} \frac{v}{k} = c, \]
which avoids contradicting the principle that nothing can travel faster than light-light. To see that the two sets of equation are compatible, let \( v = kc \), then
\[ x' = \frac{x - \frac{v}{k} t}{\sqrt{1 - \frac{v^2}{k^2 c^2}}} \rightarrow x' = \frac{x - \frac{k}{k} c t}{\sqrt{1 - \frac{k^2 c^2}{k^2 c^2}}} = \frac{x - ct}{\sqrt{1 - \frac{k^2 c^2}{k^2 c^2}}} \]

To see the time factor, recall that in the light-light case
\[ t' = \gamma \left( t - \frac{v x}{c^2} \right) = \gamma \left( t - \frac{\beta x}{c} \right), \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}}, \quad \beta = \frac{v}{c} \]

In the dark-light case, \( \beta \) must be replaced by \( \beta/k \). Hence,
\[ t' = \gamma \left( t - \frac{v x}{k c^2} \right) = \gamma \left( t - \frac{\beta x}{k c} \right), \quad \gamma = \frac{1}{\sqrt{1 - \left( \frac{\beta}{k} \right)^2}}, \quad \frac{\beta}{k} = \frac{v}{kc} \]

The upshot is that, if the speed of an object relative to light-light is measured as \( \nu \), then, relative to dark-light, its speed is \( \nu/k \).

6.3 Time as an Instantiation

What is time? One way of developing a concept of 'time' is by imagining an external world moving in time relative to a stationary observer in space-time. A concept of time emerges by comparing the movement of time in the external world to a stationary point in space-time. It is assumed that a spatial separation exists between the stationary space-time observer and the moving “in time” external world.
Suppose the observer, stationary in space-time, creates a device for measuring 'external time' (a clock) (see fig. 6.3-1), believing there is an 'intrinsic time' in the external world. The device ticks off a certain number of time increments labeled \( \Delta t \) and can be adjusted so more or less \( \Delta t \) ticks off in a given amount of time. Under this scenario, the \( \Delta t \) are constant. The number of \( \Delta t \) the device ticks off is given by \( v \Delta t \), where a greater \( v \) signifies a higher number of ticks. The product \( t = v \Delta t \) is called 'arbitrary time'. The observer stationary in time wishes to synchronize the clock with 'external time', which is signified by \( \bar{t} = \bar{v} \Delta \bar{t} \). The observer stationary in space-time must adjust the clock so that \( t = \bar{t} \), where \( \bar{v} \) signifies the 'rate of external time' measured in time increments of \( \Delta \bar{t} \). Hence, \( \bar{t} \) instantiates \( t \). To accomplish the synchronization, a second clock is constructed identical to the first, the clocks are synchronized, then one clock is given to a second observer, the other clock remaining with the observer stationary in space-time.

A light wave is sent from \( \langle 0,0 \rangle \). The instant the light wave is emitted, a second observer, located at \( \langle 0, x \rangle \) moves from \( \langle 0, x \rangle \) to \( \langle \bar{v} \Delta \bar{t}, x \rangle \) at the rate of 'external time' relative to the observer stationary in space-time. The second observer counts the number of ticks on the second clock until the light wave reaches the point \( \langle \bar{v} \Delta \bar{t}, x \rangle \).

Once the experiment concludes, the "moving in time" observer can report that \( v \) ticks were counted during the journey and the "stationary in time" observer can synchronize arbitrary time with external time i.e. \( v \Delta t = \bar{v} \Delta \bar{t} \). From fig. 6.3-1,

\[
\left( \frac{x}{c} \right)^2 + \left( \frac{\bar{v}}{c} \Delta t \right)^2 = (\Delta t)^2 \rightarrow \frac{x^2}{c^2} = \left( 1 - \frac{v^2}{c^2} \right) \Delta t^2 \rightarrow \Delta \bar{t} = \sqrt{ \frac{1 - v^2}{c^2} } \Delta t \rightarrow \Delta \bar{t} = \frac{\Delta \bar{t}}{\sqrt{ 1 - \frac{v^2}{c^2} }},
\]

\[
\bar{t} = \frac{x}{c},
\]

Note that, in the scenario above, \( \bar{t} = x/c \), where \( c, x \) are constants, but \( x \) is arbitrary. Hence, \( v \) is a function of \( \bar{t} \) i.e. \( v = \bar{v}(\bar{t}) \), since a different choice of \( x \) induces a different \( v \). The rate '\( v \)' is given units 'cm/sec'. But keep in mind that 'cm', as represented here,
is a distance in ‘time’, not a distance in ‘space’. In essence, \( v \) represents how fast time is moving relative to the speed of light. The quantity ‘\( \nabla t \)’ is called the ‘intrinsic time’ interval and \( \nabla \tilde{t} \) is called the ‘clock time’ interval.

Suppose that \( \tilde{v}(\tilde{t}) = 0 \), then
\[
\nabla t = \nabla \tilde{t}
\]

In this case, time increments are arbitrary. In other words, there is no ‘intrinsic time’. Time is arbitrary. On the other hand, suppose
\[
\tilde{v}(\tilde{t}) = K, \quad K = a \text{ constant}
\]

If
\[
\frac{1}{\sqrt{1 - \frac{K^2}{c^2}}} = k, \quad k = a \text{ constant} \rightarrow \nabla t = k \nabla \tilde{t}, \quad \tilde{t} = \frac{x}{c},
\]

Since \( x \) is arbitrary, it is only necessary to replace \( x \) with \( x/k \) to achieve
\[
\nabla t = k \frac{x}{c} = \frac{x}{c} = \nabla \tilde{t}
\]

Again, in this sense, time increments are arbitrary. There is no ‘intrinsic time’. In other words, according to the principle of relativity, an observer in space-time cannot distinguish between being at rest in space-time and moving in time at a constant rate.

**6.3.1 The Modern View of Time**

In the modern view, time is synonymous with extension (distance). Since \( x = vt \), if \( v = c \), then
\[
x = ct \rightarrow x = t, \quad c = 1
\]

and
\[
\frac{1}{c} \int_0^x dx = \int_0^\tau d\tilde{t} = \int_0^t \sqrt{1 - \frac{v^2}{c^2}} dt = \sqrt{1 - \frac{v^2}{c^2}} \int_0^t dt
\]

The proper-time ‘\( \tau \)’, in modern physics, is interpreted as a ‘distance’ i.e.
\[
\int_0^\tau d\tau = \tau
\]

where \( \tau \) represents the distance along a worldline and is invariant. All observers will agree on the proper-time. Note that, in this case, ‘\( v \)’ represents ‘spatial velocity’ and not the velocity of ‘time’.
6.3.2 Intrinsic Time

In the scenarios above, there is no 'intrinsic time'. In these cases, either time is synonymous with distance or it is an arbitrary measure. In order for time to be an intrinsic part of Nature, the rate of time \( v = \bar{v}(\bar{t}) \) must be a non-constant function of \( \bar{t} \), where \( \bar{t} \) signifies the 'clock time'. The simplest conjecture for an intrinsic time is that the rate of time \( \bar{v}(\bar{t}) \) is linear i.e.

\[
\frac{d\bar{v}}{d\bar{t}} = C \rightarrow \int d\bar{v} = \int C \ d\bar{t} \rightarrow \bar{v}(\bar{t}) = C\bar{t} + K, \quad C, K = \text{a constant}
\]

6.3.2.1 Conjecture on the Rate of Intrinsic Time

As a matter of hypothesis, let

\[
\bar{v}(\bar{t}) = [l]G\bar{t},
\]

where \([l] = 1[cm/sec^2]\) has been inserted as a factor, since \( G \) is dimensionless. If there is a constant acceleration in the direction of \( \bar{t} \) with magnitude '\( G \)', then

\[
dt = \frac{d\bar{t}}{\sqrt{1 - \frac{\bar{v}(\bar{t})^2}{c^2}}} \rightarrow dt = \frac{d\bar{t}}{\sqrt{1 - \frac{[l]^2G^2\bar{t}^2}{c^2}}}, \quad \nabla t \rightarrow dt, \quad \nabla \bar{t} \rightarrow d\bar{t},
\]

where \( t \) is the 'intrinsic time' and \( \bar{t} \) is the 'clock time'.

Let

\[
\gamma = \frac{1}{\sqrt{1 - \frac{[l]^2G^2\bar{t}^2}{c^2}}} \rightarrow dt = \gamma d\bar{t} \rightarrow \int dt = \int \gamma d\bar{t} = \int \gamma \left(\frac{x}{c}\right) d\left(\frac{x}{c}\right), \quad \bar{t} = \frac{x}{c}
\]

Hence,

\[
\int \frac{dx}{\sqrt{1 - \frac{[l]^2G^2x^2}{c^4}}} = c \int dt
\]

A constant acceleration in the direction of intrinsic time would cause space to expand. An observer at rest would witness space expanding at an accelerated rate in all directions. Note that \( G^2/c^2 \) is an extremely small number, on the order of \( \approx 10^{-50} \). So, if \( \bar{t} \) is small, then \( dt/d\bar{t} \approx 1 \).

Moreover, there is a 'clock time limit' on the order of

\[
\bar{t} \approx \sqrt{10^{50}} [\text{sec}],
\]

in which case, if \( \bar{t} > \sqrt{10^{50}} \), intrinsic time would become imaginary.
Note that the proper-time ‘τ’ would no longer be invariant. The symmetry between τ and t is broken by the acceleration ‘G’ in the direction of time.

Interestingly, if this conjecture is true, it would take about 1/3 of a second for the Universe to expand to a diameter of approximately $10^{10} [cm]$, which is about 62,137 miles. To put things in perspective, that is about 7% the diameter of the Sun or roughly 7.7 Earth diameters. However, to expand to a size of approximately $10^{20} [cm]$ across, which is about $6.2 \times 10^{14}$ miles or about 106 light-years across, would take approximately 106 years. It would take about $10^8$ billion years for the Universe to expand to a size of approximately $10^{17}$ light-years across at which time the universe would cease expansion.

### 6.4 Dark- and Light-Proper-Times

Recall from the principle of relativity that

\[ c^2 t^2 - x^2 = (ct')^2 - (x')^2, \]

if the analysis is limited to one spatial dimension. It is always possible to choose a coordinate system such that $x' = 0$, in which case,

\[ c^2 t^2 - x^2 = (ct')^2 \]

Dividing through by $c^2$ gives

\[ t^2 - \frac{x^2}{c^2} = t'^2 \rightarrow \tau^2 = t^2 - x^2, \quad c = 1, \quad \tau = t', \]

where τ is called the ‘proper-time’. Since τ is invariant, all observers will agree on the proper-time. In four-dimensional space,

\[ \tau^2 = t^2 - \left( \frac{x}{c} \right)^2 - \left( \frac{y}{c} \right)^2 - \left( \frac{z}{c} \right)^2 \rightarrow \tau^2 = t^2 - x^2 - y^2 - z^2, \quad c = 1 \]

Image theory predicts a dark- ‘τ_D’ and a light-proper-time ‘τ_L’. Proper-time is usually expressed in the form of a Lorentz invariant 4-vector. For instance,

\[ \tau_L = (t, i \frac{x}{c}, i \frac{y}{c}, i \frac{z}{c}) \rightarrow \tau_L = (t, ix, iy, iz), \quad c = 1 \]

\[ \tau_D = (t, i \frac{x}{kc}, i \frac{y}{kc}, i \frac{z}{kc}) \rightarrow \tau_D = (t, i \frac{x}{k}, i \frac{y}{k}, i \frac{z}{k}), \quad c = 1 \]
Note that

\[ \tau_L - \tau_D = \tau_{L-D} = \langle t, ix, iy, iz \rangle - \left( t, \frac{x}{k}, \frac{y}{k}, \frac{z}{k} \right) = \left( 0, i \left( 1 - \frac{1}{k} \right) x, i \left( 1 - \frac{1}{k} \right) y, i \left( 1 - \frac{1}{k} \right) z \right) = \left( \frac{k - 1}{k} \right) \langle 0, ix, iy, iz \rangle \]

If two events happen simultaneously, as calculated by the two proper-times, there will be a difference in spatial location. This agrees with a previous result, which showed that if two observers, one measuring relative to light-light, the other measuring relative to dark-light, agree on the time of an event, they will disagree on the event’s spatial location.

Moreover,

\[ \langle \tau_L | \tau_L \rangle = \tau_L^2 = t^2 - \left( \frac{x}{c} \right)^2 - \frac{y^2}{c^2} - \frac{z^2}{c^2} , \quad \langle \tau_D | \tau_D \rangle = \tau_D^2 = t^2 - \left( \frac{x}{c'} \right)^2 - \frac{y^2}{c'^2} - \frac{z^2}{c'^2} \]

Since both \( \tau_D \) and \( \tau_L \) are invariant, their difference \( '\tau_L - \tau_D' \) is invariant. If the problem is restricted to one space and one time dimension, then

\[ \tau_L^2 = t^2 - \left( \frac{x}{c} \right)^2 , \quad \tau_D^2 = t^2 - \left( \frac{x}{c'} \right)^2 \]

Therefore,

\[ \tau_D^2 - \tau_L^2 = \tau_{D-L}^2 = - \left( \frac{x}{c'} \right)^2 + \left( \frac{x}{c} \right)^2 = \left( - \left( \frac{1}{c'} \right)^2 + \left( \frac{1}{c} \right)^2 \right) x^2 \rightarrow \tau_{D-L}^2 = \left( - \left( \frac{1}{kc} \right)^2 + \left( \frac{1}{c} \right)^2 \right) x^2 \]

\[ = \left( - \left( \frac{1}{k} \right)^2 + 1 \right) x^2 = \left( \frac{k^2 - 1}{k^2} \right) x^2 , \quad c' = kc , \quad c = 1 \]

### 6.5 Dark and Light 4-Velocities

Suppose a particle is moving along a worldline. If the worldline is divided up into tiny segments, then

\[ \langle c \Delta x^0, \Delta x^1, \Delta x^2, \Delta x^3 \rangle = \Delta x^\mu_L, \quad \mu = 0, \ldots, 3, \]

where \( \Delta x^\mu_L \) represents the space-time distance between two adjacent world-points. According to image theory, a worldline can be divided into tiny segments based on the speed of dark-light i.e.

\[ \langle c' \Delta x^0, \Delta x^1, \Delta x^2, \Delta x^3 \rangle = \Delta x^\mu_D, \quad \mu = 0, \ldots, 3 \]

Now

\[ \Delta x^\mu_D - \Delta x^\mu_L = \Delta x^\mu_{D-L} = \langle c' \Delta x^0, \Delta x^1, \Delta x^2, \Delta x^3 \rangle - \langle c \Delta x^0, \Delta x^1, \Delta x^2, \Delta x^3 \rangle = \langle (c' - c) \Delta x^0, 0, 0, 0 \rangle = \langle (kc - c) \Delta x^0, 0, 0, 0 \rangle = (k - 1) \langle \Delta x^0, 0, 0, 0 \rangle, \quad c = 1 \]
If the spatial increments are the same, the time increment will be different. In this case, the discrepancy between the light-light space-time distance and dark-light space-time distance will arise only in the time dimension. The ratio of this difference is equal to a constant i.e.

\[
\frac{\Delta x^0_D - \Delta x^0_L}{\Delta x^0} = k - 1
\]

Remembering that

\[
\tau^2_{D-L} = \left(\frac{k^2 - 1}{k^2}\right)(x^1)^2 + (x^2)^2 + (x^3)^2 \rightarrow \Delta \tau_{D-L} = \lambda \sqrt{(\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2},
\]

\[
\lambda = \frac{\sqrt{(k^2 - 1)/k^2}}{c}, \quad c = 1
\]

Recall that \( U^\mu = \Delta x^\mu / \Delta \tau, \mu = 0, \ldots, 3 \) is the '4-velocity' in space-time and is invariant if it is specified in terms of the proper-time. Hence,

\[
U^\mu_{D-L} = \frac{\Delta x^\mu_{D-L}}{\Delta \tau_{D-L}} = \frac{(k - 1)(\Delta x^0, 0, 0, 0)}{\lambda \sqrt{(\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2}}, \quad \mu = 0, \ldots, 3
\]

Letting \( \Delta \rightarrow d \), then

\[
U^\mu_{D-L} = \frac{(k - 1)dt}{\lambda \sqrt{(dx^1)^2 + (dx^2)^2 + (dx^3)^2}} = \frac{k - 1}{\lambda v} = \frac{k - 1}{\lambda v} \rightarrow U^\mu_{D-L}
\]

\[
= \frac{(k - 1)c^2}{\lambda v}, \quad dx^0 = dt, \quad v = \sqrt{\frac{(dx^1)^2}{dt^2} + \frac{(dx^2)^2}{dt^2} + \frac{(dx^3)^2}{dt^2}} > 0,
\]

where \( c \) has been reinserted for dimensional consistency. Note that \( U^\mu_{D-L} \) is a constant, and hence, invariant and that

\[
U^\mu_{D-L} = \frac{\Delta x^0_D - \Delta x^0_L}{\Delta \tau_D - \Delta \tau_L} = \frac{\Delta x^0_D - \Delta x^0_L}{\Delta \tau_D - \Delta \tau_L} \rightarrow \frac{dx^0_{D-L}}{\Delta \tau_{D-L}} = \frac{(k - 1)dt}{\Delta \tau_{D-L}}, \quad U^1_{D-L} = U^2_{D-L} = U^3_{D-L} = 0,
\]

6.6 Applying Hamilton’s Principle to Space-Time

Momentum and energy are described in terms of the proper-time ‘\( \tau \)’, since all observers agree on the proper-time. Dividing \( \tau \) into a number of small pieces ‘\( \Delta \tau \)’, then \( \sum \Delta \tau = \tau \) represents the total proper-time along the worldline. Moreover, \( \sum \Delta \tau \) is an instance of the least-action principle associated with classical mechanics i.e.

\[
\text{action} = \sum \Delta \tau
\]
The action along any trajectory is stationary (usually a minimum), so if it is multiplied by a constant, it remains stationary. In other words, if $\sum \Delta \tau$ is replaced by

$$-m \sum \Delta \tau, \quad m = a \text{ constant},$$

the stationarity of the action remains. If the worldline is continuously smooth, then

$$\lim_{\Delta \tau \to 0} \Delta \tau = d\tau$$

and

$$-m \int d\tau$$

Image theory predicts both a dark-‘$\tau_D$’ and a light-proper-time ‘$\tau_L$’:

$$-m \int d\tau_L = -m \int \sqrt{dt^2 - \frac{dx^i dx_i}{c^2}} = -m \int \sqrt{1 - \frac{x^i \dot{x}_i}{c^2}} dt,$$

$$dx^i dx_i = dx^2 + dy^2 + dz^2$$

Likewise,

$$-m \int d\tau_D = -m \int \sqrt{dt^2 - \frac{dx^i dx_i}{k^2 c^2}} = -m \int \sqrt{1 - \frac{x^i \dot{x}_i}{k^2 c^2}} dt,$$

The integrand ‘$\sqrt{1 - \dot{x}^i \dot{x}_i/c^2}$’ is simply a function of the velocity. The integrals are of the form

$$\int L_L dt, \quad L_L = -m \sqrt{1 - \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{c^2}}, \quad \int L_D dt, \quad L_D = -m \sqrt{1 - \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{k^2 c^2}},$$

where $L$ signifies a ‘Lagrangian density function’. Let $c = 1$, then

$$L_L = -m \sqrt{1 - v^2},$$

which can be approximated by the binomial theorem:

$$(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)x^2}{2!} \pm \frac{n(n-1)(n-2)x^3}{3!} + \ldots \to -m(1 - v^2)^{1/2}$$

$$\approx -m \left(1 - \frac{1}{2} v^2\right), \quad x = v^2, \quad n = \frac{1}{2},$$

where higher terms beyond the second are discarded as too small to count.
On the other hand,

\[ L_D = -m \sqrt{1 - \frac{v^2}{k^2}} \approx -m \left(1 - \frac{1}{2} \frac{v^2}{k^2}\right), \quad x = \frac{v^2}{k^2}, \quad n = \frac{1}{2} \]

Hence,

\[ L_L = -m \sqrt{1 - v^2} \approx -m \left(1 - \frac{1}{2} v^2\right) = -m + \frac{1}{2} mv^2, \]

\[ L_D = -m \sqrt{1 - \frac{v^2}{k^2}} \approx -m \left(1 - \frac{1}{2} \frac{v^2}{k^2}\right) = -m + \frac{mv^2}{2k^2} \]

Recall the definition of momentum:

\[ p_i = \frac{\partial L}{\partial \dot{x}^i} \]

It then follows

\[ \frac{\partial L_L}{\partial \dot{x}} = p_{Lx} = \frac{\partial \left(-m \sqrt{1 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2}\right)}{\partial \dot{x}} = \frac{m\dot{x}}{\sqrt{1 - v^2}} \]

\[ \frac{\partial L_D}{\partial \dot{x}} = p_{Dx} = \frac{\partial \left(-m \sqrt{1 - \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{k^2}}\right)}{\partial \dot{x}} = \frac{m\dot{x}}{k^2 \sqrt{1 - \frac{v^2}{k^2}}} \]

Likewise,

\[ \frac{\partial L_L}{\partial \dot{y}} = p_{Ly} = \frac{\partial \left(-m \sqrt{1 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2}\right)}{\partial \dot{y}} = \frac{m\dot{y}}{\sqrt{1 - v^2}} \]

\[ \frac{\partial L_D}{\partial \dot{y}} = p_{Dy} = \frac{\partial \left(-m \sqrt{1 - \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{k^2}}\right)}{\partial \dot{y}} = \frac{m\dot{y}}{k^2 \sqrt{1 - \frac{v^2}{k^2}}} \]

\[ \frac{\partial L_L}{\partial \dot{z}} = p_{Lz} = \frac{\partial \left(-m \sqrt{1 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2}\right)}{\partial \dot{z}} = \frac{m\dot{z}}{\sqrt{1 - v^2}} \]

\[ \frac{\partial L_D}{\partial \dot{z}} = p_{Dz} = \frac{\partial \left(-m \sqrt{1 - \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{k^2}}\right)}{\partial \dot{z}} = \frac{m\dot{z}}{k^2 \sqrt{1 - \frac{v^2}{k^2}}} \]
Recalling that
\[ U_L^x = U_L^1 = \frac{v^1}{\sqrt{1 - v^2}} \]
then
\[ p_{Lx} = mU_L^1 \]
Hence,
\[ p_{Lx} = mU_L^1, \quad p_{Ly} = mU_L^2, \quad p_{Lz} = mU_L^3 \rightarrow p_{Li} = mU^i, \quad i = 1, \ldots, 3, \]
Similarly,
\[ U_D^x = U_D^1 = \frac{v^1}{k^2 \sqrt{1 - \frac{v^2}{k^2}}} \]
\[ p_{Dx} = mU_D^1, \quad p_{Dy} = mU_D^2, \quad p_{Dz} = mU_D^3 \rightarrow p_{Di} = mU_D^i \]

**6.6.1 Light and Dark Energy**

In the Hamiltonian formulation of the energy
\[ H = \sum_i \dot{x}_i p_i - \mathcal{L}, \]
where \( H \) represents the ‘energy’ of the system. By letting
\[ p_{Li} = \frac{m \dot{x}_i}{\sqrt{1 - v^2}}, \quad \mathcal{L}_L = -m \sqrt{1 - v^2}, \]
then
\[ H_L = \sum_i \dot{x}_i \frac{m \dot{x}_i}{\sqrt{1 - v^2}} + m \sqrt{1 - v^2} = \sum_i \frac{m \dot{x}_i^2}{\sqrt{1 - v^2}} + m \frac{1 - v^2}{\sqrt{1 - v^2}} = \frac{mv^2}{\sqrt{1 - v^2}} + \frac{m}{\sqrt{1 - v^2}} \]
\[ \quad = \frac{m}{\sqrt{1 - v^2}} \]
Similarly,
\[ H_D = \frac{m}{\sqrt{1 - \left(\frac{v}{k}\right)^2}} \]
But

\[ U_L^0 = \frac{1}{\sqrt{1 - v^2}} \rightarrow p_{L0} = \frac{m}{\sqrt{1 - v^2}} \]

and, by the binomial theorem,

\[ \frac{m}{\sqrt{1 - v^2}} \approx m + \frac{mv^2}{2} \rightarrow \frac{m}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} \approx m + \frac{mv^2}{2c^2} \]

By the same token,

\[ U_D^0 = \frac{1}{k^2 \sqrt{1 - \frac{v^2}{k^2}}} \rightarrow p_{D0} = \frac{m}{k^2 \sqrt{1 - \frac{v^2}{k^2}}} \]

Hence,

\[ \frac{m}{k^2 \sqrt{1 - \frac{v^2}{k^2}}} \approx \frac{m}{k^2} + \frac{mv^2}{2k^4} \rightarrow \frac{m}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = m + \frac{mv^2}{2k^2} = \frac{m}{\sqrt{1 - \frac{v^2}{k^2c^2}}} = m + \frac{mv^2}{2k^2c^2} \]

Multiplying through by \( c^2 \) and \((c')^2\), the equations above become respectfully

\[ \frac{mc^2}{\sqrt{1 - \left(\frac{v}{c'}\right)^2}} \approx mc^2 + \frac{mv^2}{2} \],

\[ \frac{mc'^2}{\sqrt{1 - \left(\frac{v}{c'}\right)^2}} = mc'^2 + \frac{mv^2}{2} \],

which have dimensions of energy.

Let \( E_L = mc^2/\sqrt{1 - (v/c)^2} \), then

\[ E_L = mc^2 + \frac{mv^2}{2} \rightarrow E_L = mc^2, \quad \text{if } v = 0 \]

Likewise, letting

\[ E_D = -\frac{m(c')^2}{\sqrt{1 - \left(\frac{v}{c'}\right)^2}} \]
then

\[ E_D = m(c')^2 + \frac{mv^2}{2} \rightarrow E_D = m(c')^2, \quad \text{if } v = 0 \]

Recall from special relativity that if \( c = 1 \), then

\[(U_L^0)^2 - U_L^i U_L^i = 1 \rightarrow m^2 = (mU_L^0)^2 - p_L^2 = p_L^2 - p_L^2 \]

Moreover,

\[ U_L^0 = \frac{1}{\sqrt{1 - v^2}} \rightarrow mU_L^0 = \frac{m}{\sqrt{1 - v^2}} \]

Hence,

\[ m^2 = \left( \frac{m}{\sqrt{1 - v^2}} \right)^2 - p_L^2 - p_L^2 - p_L^2 \]

To make the equation above dimensionally consistent, write

\[(mc^2)^2 = \left( \frac{mc^2}{\sqrt{1 - v^2/c^2}} \right)^2 - c^2 p_L^2 - c^2 p_L^2 - c^2 p_L^2 \rightarrow m^2 c^4 = E_L^2 - p_L^2 c^2 \rightarrow E_L^2 \]

\[ = p_L^2 c^2 + m^2 c^4 \rightarrow E_L = \sqrt{p_L^2 c^2 + m^2 c^4} \]

Likewise,

\[ E_D = \sqrt{p_D^2 (c')^2 + m^2 (c')^4}, \quad c' = kc \]

### 6.6.2 The Difference between Light and Dark Energy

Recall that

\[ U_{D-L}^\mu \frac{dx_{D-L}^\mu}{d\tau_{D-L}} = \frac{(k - 1)(dt, 0, 0, 0)}{\lambda \sqrt{(dx^1)^2 + (dx^2)^2 + (dx^3)}} = \frac{(k - 1)c}{\lambda v} = \frac{(k - 1)c^2}{\lambda v} \rightarrow (U_{D-L}^\mu)^2 \]

\[ \frac{(k - 1)c^4}{\lambda^2 v^2} \rightarrow E_{D-L} = \frac{(k - 1)c^4}{\lambda^2 v^2}, \quad \mu = 0,...,3, \quad \lambda^2 = \frac{k^2 - 1}{k^2}, \quad |v| > 0, \]

where \( E_{D-L} \) represents the difference between the dark- and light-energy.
6.7 Classical Field Theory and Space-Time

A ‘field’ is a function \( \varphi(t, x) \) that gives a value for \( \varphi \) at every point in space-time. Classical and relativistic mechanics assume that the world is governed by the principle of ‘least action’ – that trajectories, regardless of the number of spatial dimensions, move so as to minimize the action.

If \( \varphi = \varphi(t, x^i) \) is a function of one time and three spatial dimensions, the action ‘\( A \)’ is

\[
A = \int L \, dt \, dx \, dy \, dz = \int L \, dx^\mu, \quad dx^0 = dt, \quad dx^1 = dx, \quad dx^2 = dy, \quad dx^3 = dz,
\]

where

\[
L = L(\varphi, \frac{\partial \varphi}{\partial x^\mu}), \quad \mu = 0, \ldots, 3
\]

The equations of motion are

\[
\frac{\partial}{\partial x^\mu} \left( \frac{\partial L}{\partial \left( \frac{\partial \varphi}{\partial x^\mu} \right)} \right) - \frac{\partial L}{\partial \varphi} = 0
\]

6.7.1 Dark- and Light-Light Waves

A Lagrangian ‘\( L \)’ in 4-dimensional space-time might look something like

\[
L = \frac{m}{2} \left[ \frac{\left( \frac{\partial \varphi}{\partial t} \right)^2}{c^2} - \frac{\left( \frac{\partial \varphi}{\partial x} \right)^2}{c^2} - \frac{\left( \frac{\partial \varphi}{\partial y} \right)^2}{c^2} - \frac{\left( \frac{\partial \varphi}{\partial z} \right)^2}{c^2} \right] - V(\varphi)
\]

Note that

\[
\frac{\partial L}{\partial x^0} = \frac{\partial L}{\partial t} = m \dot{\varphi} \rightarrow \frac{\partial \left( \frac{\partial L}{\partial t} \right)}{\partial t} = m \ddot{\varphi}
\]

Likewise,

\[
\frac{\partial L}{\partial x^i} = m \frac{\partial \varphi}{\partial x^i} \rightarrow \frac{\partial}{\partial x^i} \left( \frac{\partial L}{\partial x^i} \right) = m \frac{\partial^2 \varphi}{(\partial x^i)^2}, \quad i = 1, \ldots, 3
\]

Hence,

\[
\frac{\partial}{\partial x^\mu} \left( \frac{\partial L}{\partial \left( \frac{\partial \varphi}{\partial x^\mu} \right)} \right) - \frac{\partial L}{\partial \varphi} = m \left[ \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial y^2} - \frac{\partial^2 \varphi}{\partial z^2} \right] + \frac{\partial V}{\partial \varphi} = 0,
\]

where \( 1/c^2 \) has been inserted for dimensionally consistency. These are the equations of motion in light-space.
Likewise, there are equations of motion in dark-space:

\[
\frac{\partial}{\partial x^\mu} \left( \frac{\partial L}{\partial (\nabla \phi)} \right) - \frac{\partial L}{\partial \phi} = m \left[ \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \phi}{\partial z^2} \right] + \frac{\partial V}{\partial \phi} = 0
\]

If \( \partial V / \partial \phi = 0 \) and the \( y, z \)-dimensions ignored, then the expressions in the square brackets become respectively

\[
\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = 0, \quad \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = 0, \quad c' = kc
\]

The equations above convey the motion of a free wave traveling in the direction of the \( x \)-axis. If dark-light comes in the form of a wave, then

\[
\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = 0 \rightarrow (k^2 - 1) \frac{\partial^2 \phi}{\partial t^2} = 0 \rightarrow \phi = Ct + K, \quad c = 1, \quad C, K = \text{a constant},
\]

suggesting that the difference between the motion of a free moving light-light and a free moving dark-light wave is linear in time, which is not too surprising, since the difference in speeds of the two waves is a constant. Evidently, \( C = (k - 1)c \), the difference between the speeds of the two light waves.

**6.7.2 Harmonic Oscillators**

Of course, \( \partial V / \partial \phi \) does not always equal zero. Suppose

\[
V_L(\phi) = (\mu^2 / 2)\phi^2, \quad V_D(\phi) = (\omega^2 / 2)\phi^2,
\]

then

\[
\partial V_L / \partial \phi = \mu^2 \phi, \quad \partial V_D / \partial \phi = \omega^2 \phi
\]

The wave equations become

\[
\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + \mu^2 \phi = 0, \quad \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + \omega^2 \phi = 0
\]

respectfully. Hence,

\[
\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + \mu^2 \phi - \left( \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + \omega^2 \phi \right) = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} + \mu^2 \phi - \omega^2 \phi
\]

\[
= \left( \frac{k^2 - 1}{k^2} \right) \frac{\partial^2 \phi}{\partial t^2} + (\mu^2 - \omega^2)\phi = 0 \rightarrow \frac{\partial^2 \phi}{\partial t^2} + (\mu^2 - \omega^2)\phi = 0, \quad c = 1,
\]

\[
(k^2 - 1) / k^2 \approx 1
\]
The last equation is the equation of a harmonic oscillator, where \((\mu^2 - \omega^2)\) represents the spring constant.

### 6.8 Proper vs. Intrinsic Time

The concept of ‘proper-time’ emerges from the principle of relativity. Recall that the principle of relativity is expressed mathematically by

\[
x_L^2 + y_L^2 + z_L^2 - c^2 t^2 = (x'_L)^2 + (y'_L)^2 + (z'_L)^2 - (ct')^2
\]

It is always possible to choose coordinates such that \(x'_L, y'_L, z'_L = 0\). The equation above then reduces to

\[
x_L^2 + y_L^2 + z_L^2 - c^2 t^2 = -(ct')^2 \rightarrow (t')^2 = t^2 - \frac{1}{c^2}(x_L^2 + y_L^2 + z_L^2)
\]

\[
v^2 = \frac{x_L^2}{t^2} + \frac{y_L^2}{t^2} + \frac{z_L^2}{t^2},
\]

where \(\tau\) is called the ‘proper-time’ and \(t\) is called the ‘coordinate time’. Note that the proper-time is invariant. All observers will agree on the proper-time. If \(v = 0\), then the proper-time is equal to the coordinate time.

#### 6.8.1 Intrinsic Proper-Time

Recall that in Sec. 6.3.2, it was shown that Nature has an ‘intrinsic time’ only if the rate of time, designated ‘\(\ddot{v}(\bar{t})\)’, is a non-constant function of \(\bar{t}\). Moreover, the previous chapter (see Secs. 5.4/5.5) revealed a missing acceleration factor somewhere in the physics, although no discussion ensued concerning what the missing acceleration factor entailed. Consider the statement ‘\(v = c\)’. Note that the dimensions associated with this statement are \([L/T]\). Let \(a = [L]\), where \([L] = [L/T^2]\) and \(G\) is dimensionless, then

\[
\frac{v}{a} = \frac{c}{G}[T],
\]

where \(v/a\) has dimensions ‘[T]’ (time). Hence, the conjecture is that the missing acceleration factor emerges within the time dimension. Let

\[
\frac{\ddot{v}(\bar{t})}{v} = \frac{a \bar{t}}{v} \rightarrow \left(\frac{a \bar{t}}{v}\right)^2 = \frac{G^2 \bar{t}^2}{c^2}, \quad a = G, \quad v = c
\]

is a dimensionless expression that incorporates the missing acceleration factor. The simplest idea for an intrinsic time is to set the rate of time ‘\(\ddot{v}(\bar{t})\)’ as linear in time i.e.

\[
\frac{d\ddot{v}}{d\bar{t}} = C \rightarrow \int d\ddot{v} = \int C \, d\bar{t} \rightarrow \ddot{v}(\bar{t}) = C\bar{t} + K, \quad C, K = \text{a constant}
\]
If $\vec{v}(\vec{t}) = [l]G\vec{t}$, then there is a constant acceleration in the direction of $\vec{t}$ with magnitude ‘$G$’. It then follows that the intrinsic time element ‘$dt$’ is designated

$$dt = \frac{d\vec{t}}{\sqrt{1 - \frac{[l]^2G^2t^2}{c^2}}}$$

which contains the missing acceleration factor.

The proper-time element is given by

$$d\tau^2 = \left(1 - \frac{v^2}{c^2}\right)dt^2$$

If $v = 0$, then the proper-time is equal to the coordinate time. Moreover,

$$\frac{d\tau^2}{dt^2} = 1 \rightarrow \frac{d}{dt}\left(\frac{d\tau}{dt}\right) = 0$$

However, let

$$(d\tau')^2 = \gamma^2\left(1 - \frac{v^2}{c^2}\right)dt^2, \quad \gamma = \frac{1}{\sqrt{1 - \frac{[l]^2G^2t^2}{c^2}}}$$

where $\tau'$ is called the ‘intrinsic proper-time’. If $v = 0$, then

$$(d\tau')^2 = \gamma^2dt^2 \rightarrow d\tau' = \gamma dt$$

In this case, the intrinsic proper-time is equal to the coordinate time if and only if $t = 0$. Note that

$$\frac{d}{dt}\left(\frac{d\tau'}{dt}\right) = \frac{d\gamma}{dt} = \frac{G^2t}{c^2\left(1 - \frac{G^2t^2}{c^2}\right)^{3/2}}$$

which is not, in general, equal to zero. The upshot is that the intrinsic proper-time is not invariant. Replacing the proper-time with intrinsic proper-time in Einstein’s equations creates issues within the general theory of relativity. The issues will be addressed in the next chapter.

6.9 Concluding Remarks

The theory of relativity within image theory emerges from ‘instantiation’. The complex pair ‘$(x,y)$’ can be instantiated by

$$\left(\sqrt{x}\sqrt{x + y}, \sqrt{y}\sqrt{x + y}\right)$$
but also by
\[ \langle \sqrt{y}, \sqrt{x+y}, -\sqrt{x}, \sqrt{x+y} \rangle \]

The principle of relativity implies that these instantiations are equivalent. According to this principle, two observers moving relative to each other at a constant speed have equivalent perspectives. In other words, physical laws written from either perspective must agree, even though the two observers will not agree on the location or time of an event in space-time. In the context of special relativity, time and space must be conceived as one entity, since the rate at which time passes depends on the relative velocity through space.

Position in ordinary space is specified by three dimensions. In the Cartesian coordinate system, these dimensions are called ‘\( x \)’, ‘\( y \)’, and ‘\( z \)’. A position in space-time is called an ‘event’, but its specification requires four numbers, the three spatial dimensions plus one ‘time’ dimension. An ‘event’ is something that happens instantaneously at a single point in space-time, represented by a set of coordinates \( \langle x, y, z, t \rangle \).

The word ‘event’ used in relativity means something different than the word ‘event’ used in normal conversation, where an ‘event’ might refer to a concert, sporting event or a battle, which all have finite durations. The word ‘event’ in space-time is often referred to as a ‘mathematical event’, which has zero duration and represents a single point in spacetime. The path of a particle through spacetime can be thought of as a succession of events. The series of events linked together forms a line called a ‘worldline’ [275].

Image theory predicts the existence of dark-light, which moves faster than light-light, but is not directly detectable. The difference in the speeds of dark- vs. light-light creates an anomaly. The theory of relativity predicts that a stationary clock will measure the rate of time passage on a clock moving at the speed of light-light as zero. But if dark-light travels faster than light-light, a clock moving with the speed of dark-light would, intuitively, run in reverse as measured by a stationary clock. However, the idea of ‘time reversal’ leads, not only to conflicts with the principle of relativity, but known empirical facts. The anomaly is overcome by recognizing that ‘\( \nu \)’, representing the velocity of an object relative to \( c \), will measure \( \nu/k \) relative to dark-light speed. This removes the conflict and leads to two sets of Lorentz transformations, one set for light-light and the other for dark-light.

A concept of ‘time’ was developed by imagining an external world moving in time relative to a stationary observer in space-time. The analysis infers that ‘intrinsic time’ exists only if the rate of intrinsic time is not a constant, which leads to the realization that intrinsic time is synonymous with spatial expansion. Evidently, space is expanding at an accelerated rate in all directions. If the hypothesis about time is true, the proper-time will no longer be invariant, which impacts directly the general theory of relativity, the consequences of which will be addressed in the next chapter.

Within the theory of relativity, proper-time is invariant. If a space-time event is determined in terms of the proper-time of light-light and also in terms of the proper-time of dark-light, if such an event, in each case, is deemed to occur at the same time, the
two calculations will give different spatial locations. If the two calculations give the same spatial location, then there will be a difference in the time of the event.

Considered individually, both the light-proper-time $\tau_L$ and the dark-proper-time $\tau_D$ are invariant. Hence, the difference between the two $\tau_{D-L}$ is also invariant. Since the 4-velocity $U^\mu$ is given in terms of the proper-time, then

$$U^\mu_{D-L} = \frac{\Delta x^\mu}{\Delta \tau_{D-L}} = \frac{\Delta x^\mu_{D-L}}{\Delta \tau_{D-L}} \rightarrow \frac{d x^\mu_{D-L}}{d \tau_{D-L}} = \frac{(k-1)dt}{d \tau_{D-L}} \rightarrow \frac{d U^\mu_{D-L}}{dt} = k-1 \frac{d \tau_{D-L}}{d \tau_{D-L}},$$

which is invariant.

According to the theory of relativity, a particle with mass $m$ has both rest and a kinetic energy i.e.

$$E_L = mc^2 + \frac{mv^2}{2} \rightarrow E_L = mc^2, \quad \text{if } v = 0,$$

where $E_L$ is the energy associated with light-light. The same can be said of dark-light:

$$E_D = m(c')^2 + \frac{mv^2}{2} \rightarrow E_D = m(c')^2, \quad \text{if } v = 0$$

Since

$$U^\mu_{D-L} = \frac{d x^\mu_{D-L}}{d \tau_{D-L}} = \frac{(k-1)(dt,0,0,0)}{\lambda \sqrt{(dx^1)^2 + (dx^2)^2 + (dx^3)^2}} = \frac{(k-1)c}{\lambda v} = \frac{(k-1)c^2}{\lambda v},$$

then

$$(U^\mu_{D-L})^2 = \frac{(k-1)^2c^4}{\lambda^2v^2} \rightarrow m(U^\mu_{D-L})^2 = \frac{(k-1)^2mc^4}{\lambda^2v^2} = E_{D-L}, \quad |v| > 0$$

Classical and relativistic mechanics maintain that the world is governed by the principle of least action. If dark-light mechanics wave-like, just like light-light, then

$$\frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} - \left[ \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} \right] = 0 \rightarrow \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} - \frac{1}{k^2c^2} \frac{\partial^2 \varphi}{\partial t^2}$$

$$= \frac{\partial^2 \varphi}{\partial t^2} - \frac{1}{k^2} \frac{\partial^2 \varphi}{\partial t^2} = \left( \frac{k^2 - 1}{k^2} \right) \frac{\partial^2 \varphi}{\partial t^2} = 0 \rightarrow \varphi(t,x) = Ct + K, \quad c = 1$$

where $C$ and $K$ are constants, suggesting that the difference between the free motion of a light-light and a dark-light wave is linear in time.
Chapter 7

Image Theory and General Relativity

“Time is an illusion.”
— Albert Einstein

7.0 Introduction

Does Nature possess an ‘intrinsic time’? Time can be measured in many different ways. It takes the Earth about $365.256$ days to orbit the Sun. The Earth rotates around its axis about every $24$ hours. Currently, the base unit of time in the International System of Units is the ‘second’ i.e.

$$1 \text{ [sec]} \approx 9 \times 10^9 \text{ [oscillations]}$$

of a cesium atom. Two synchronous cesium clocks would run for about $1,400,000$ years before their times would differ by $1 \text{ [sec]}$. However, virtually all measures of ‘time’ are arbitrary. In disagreement with the statement “time is an illusion”, the discussion in this chapter argues that Nature possesses a cosmic time.

7.0.1 Tensors as Instantiations

Suppose there is a set of ordered numbers '$x = (x^1, x^2, ..., x^n)$' called ‘coordinates’. According to the precepts of image theory, coordinates must be instantiated. If the coordinates '$\bar{x} = (\bar{x}^1, \bar{x}^2, ..., \bar{x}^n)$' are related to a second set of ordered numbers '$\bar{x} = (\bar{x}^1, \bar{x}^2, ..., \bar{x}^n)$' such that

$$x^k \to \bar{x}^k, \quad k = 1,2, ..., n,$$

then the set of coordinates '$\bar{x}$' validly instantiates the set of coordinates '$x$'.

Suppose that

$$\bar{x}^k = \bar{x}^k(x^1, x^2, ..., x^n),$$

so that the $\bar{x}^k$'s are functions of the $x^k$'s. If $x^k = x^k(\bar{x}^1, \bar{x}^2, ..., \bar{x}^n)$ such that

$$x^k \to \bar{x}^k, \quad \bar{x}^k \to x^k, \quad k = 1,2, ..., n,$$

then $\bar{x}$ validly instantiates $x$.

Suppose there is an object, in this case, a space curve ‘$C$’, described in the coordinate system ‘$x^k$’. Let $C$ be parameterized by the functions

$$x^k = x^k(t), \quad a \leq t \leq b$$
The tangent field \( T^k \) is defined
\[
T^k = \frac{dx^k}{dt}
\]
If the same curve \( C \) is described in the system \( \bar{x}^k = \bar{x}^k(t) \), then
\[
\bar{T}^k = \frac{d\bar{x}^k}{dt}
\]
Since the \( \bar{x}^k \)'s are functions of the \( x^k \)'s, then, by the chain rule of the calculus,
\[
\frac{d\bar{x}^k}{dt} = \frac{\partial \bar{x}^k}{\partial x^r} \frac{dx^r}{dt} \quad \Rightarrow \quad \bar{T}^k = \frac{\partial \bar{x}^k}{\partial x^r} T^r
\]
It is said that \( T^r \) is instantiated by \( \bar{T}^k \). In general, if \( f \) is some function of the coordinates that represents a physical quantity of interest, then consider two generic coordinate systems \( x_i \) and \( \bar{x}_i \) and assume their instantiations are known i.e.
\[
x_i = x_i(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n), \quad \bar{x}_i = \bar{x}_i(x_1, x_2, \ldots, x_n) \rightarrow x_i \mapsto \bar{x}_i
\]
If the components of the gradient of \( f \) in \( x_i \) are known, then the components of the gradient of \( f \) in \( \bar{x}_i \) can be found by employing the chain rule:
\[
\frac{\partial f}{\partial \bar{x}_i} = \frac{\partial f}{\partial x^1} \frac{\partial x^1}{\partial \bar{x}_i} + \frac{\partial f}{\partial x^2} \frac{\partial x^2}{\partial \bar{x}_i} + \cdots + \frac{\partial f}{\partial x^n} \frac{\partial x^n}{\partial \bar{x}_i} = \frac{\partial x_j}{\partial \bar{x}_i} \frac{\partial f}{\partial x^j},
\]
where repeated indices indicate a summation.
A differential \( d\bar{x}^k \) is related to a differential \( dx^k \), again by invoking the chain rule:
\[
d\bar{x}^i = \frac{\partial \bar{x}_i}{\partial x^k} dx^k,
\]
where, once more, repeated indices are summed over \([278]\). Hence,
\[
\frac{\partial f}{\partial \bar{x}_i} d\bar{x}^i = \frac{\partial x_j}{\partial \bar{x}_i} \frac{\partial f}{\partial x^j} dx^k = \frac{\partial f}{\partial x^k} dx^k
\]
However, the relationship above can be described in more general terms. For any object \( \tilde{A}^k \), subject to the condition
\[
\tilde{A}^k = \frac{\partial \bar{x}^k}{\partial x^r} A^r,
\]
then \( \tilde{A}^k \) instantiates \( A^r \). The factor \( A^r \) is called a 'contravariant tensor' of order '1', sometimes referred to as a 'contravariant vector'. Moreover, if the coefficient of \( A^r \) is inverted, then
\[ \tilde{A}_k = \frac{\partial x^r}{\partial \tilde{x}^k} A_r, \]

where the superscripts have been turned into subscripts by convention. Note that \( \tilde{A}_k \) instantiates \( A_r \). The factor ‘\( A_r \)’ is called a ‘covariant tensor’ of order ‘1’, sometimes referred to as a ‘covariant vector’. For example, let \( f \) be an arbitrary scalar function defined in the coordinate system ‘\( x^l \)’. The gradient ‘\( \nabla f \)’ of \( f \) is defined

\[ \nabla f = \left( \frac{\partial f}{\partial x^1}, \frac{\partial f}{\partial x^2}, \ldots, \frac{\partial f}{\partial x^n} \right) \rightarrow \tilde{\nabla} f = \left( \frac{\partial \tilde{f}}{\partial \tilde{x}^1}, \frac{\partial \tilde{f}}{\partial \tilde{x}^2}, \ldots, \frac{\partial \tilde{f}}{\partial \tilde{x}^n} \right) \rightarrow \frac{\partial \tilde{f}}{\partial \tilde{x}^l} = \frac{\partial f}{\partial x^r} \frac{\partial x^r}{\partial \tilde{x}^l}, \]

where \( \tilde{\nabla} f = \frac{\partial \tilde{f}}{\partial \tilde{x}^l} \) and \( T_r = \frac{\partial f}{\partial x^r} \). Hence, \( \nabla f \) is a covariant vector.

The discussion above leads to the following:

**Definition:** A ‘tensor’ is an object instantiated within all equivalent coordinate systems.

Now multiply the two equations

\[ \tilde{A}^k = \frac{\partial \tilde{x}^k}{\partial x^r} A^r, \quad \tilde{A}_k = \frac{\partial x^r}{\partial \tilde{x}^k} A_r \]

to obtain

\[ \tilde{A}^k \tilde{A}_k = \frac{\partial \tilde{x}^k}{\partial x^r} \frac{\partial x^r}{\partial \tilde{x}^k} A^r A_r \rightarrow \tilde{A}^k \tilde{A}_k = A^r A_r, \quad \frac{\partial \tilde{x}^k}{\partial x^r} \frac{\partial x^r}{\partial \tilde{x}^k} = 1, \]

where repeated indices indicate a summation. Let \( \tilde{A}^2 = \tilde{A}^k \tilde{A}_k \) and \( A^2 = A^r A_r \), then \( \tilde{A} \) not only instantiates \( A \), but \( \tilde{A} = A \), where the quantities ‘\( A \)’ and ‘\( \tilde{A} \)’ are called ‘scalars’. This leads to the following result:

**Given any contravariant vector ‘\( S^l \)’, and any covariant vector ‘\( T_i \)’, then \( S^l T_i \) is invariant.**

For example, the gradient of an arbitrary function ‘\( f \)’ is a covariant vector i.e. \( T_i = \frac{\partial f}{\partial x^i} \) and the tangent vector is a contravariant vector i.e. \( S^i = \frac{dx^i}{dt} \). Hence,

\[ T_i S^i = \frac{\partial f}{\partial x^r} \frac{dx^r}{dt} = \frac{df}{dt}, \]

so \( df / dt \) is invariant and independent of coordinate system in which the curve ‘\( x^i(t) \)’ is specified.

**7.0.1.1 Valid Instantiations of Coordinates**

Under what circumstances are coordinate systems equivalent? If

\[ \tilde{x}^k = \tilde{x}^k(x^1, x^2, \ldots, x^n), \quad i = 1, \ldots, n \]
are differentiable functions, then the partial derivatives \( \partial \bar{x}^k / \partial x^j \) arising from \( \bar{x}^k \) can be arranged in the matrix

\[
J = \begin{bmatrix}
\frac{\partial \bar{x}^1}{\partial x^1} & \frac{\partial \bar{x}^1}{\partial x^2} & \ldots & \frac{\partial \bar{x}^1}{\partial x^n} \\
\frac{\partial \bar{x}^2}{\partial x^1} & \frac{\partial \bar{x}^2}{\partial x^2} & \ldots & \frac{\partial \bar{x}^2}{\partial x^n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial \bar{x}^n}{\partial x^1} & \frac{\partial \bar{x}^n}{\partial x^2} & \ldots & \frac{\partial \bar{x}^n}{\partial x^n}
\end{bmatrix}
\]

which is called the 'Jacobian matrix' and \( \det J \) is called the 'Jacobian'. A necessary condition that a coordinate system is validly instantiated is

\[ \det J \neq 0 \]

If \( \det J \neq 0 \), then \( J \) has an inverse \( J^{-1} \) such that \( JJ^{-1} = J^{-1}J = I \). For instance, let

\[
\bar{x}^1 = x^1 x^2, \quad \bar{x}^2 = (x^2)^2,
\]

then \( \partial \bar{x}^1 / \partial x^1 = x^2, \partial \bar{x}^1 / \partial x^2 = x^1, \partial \bar{x}^2 / \partial x^1 = 0 \) and \( \partial \bar{x}^2 / \partial x^2 = 2x^2 \). Hence,

\[
J = \begin{bmatrix} x^2 & x^1 \\ 0 & 2x^2 \end{bmatrix} \rightarrow \det J = 2(x^2)^2
\]

The matrix \( J \) has an inverse so long as \( x^2 \neq 0 \). By a well-known theorem from analysis, the instantiations \( \bar{x}^1 = x^1 x^2, \bar{x}^2 = (x^2)^2 \) are locally one-to-one and onto on an open set \( \mathcal{U} \) in \( \mathbb{R}^2 \) if and only if \( \det J \neq 0 \) at each point of \( \mathcal{U} \) [200]. If, within \( \mathcal{U} \), second order derivatives exist, then \( \bar{x}^i \) is a valid instantiation of \( x^i \). Since \( J J^{-1} = J^{-1}J = I \), then

\[
\frac{\partial \bar{x}^i}{\partial x^r} \frac{\partial x^r}{\partial \bar{x}^l} = \frac{\partial x^i}{\partial x^r} \frac{\partial \bar{x}^r}{\partial x^l} = \delta^i_j
\]

### 7.0.2 Higher Order Tensors

Higher order tensors arise by multiplying lower order tensors together. For example, let

\[
\tilde{A}^k = \frac{\partial \bar{x}^k}{\partial x^r} A^r, \quad B^i = \frac{\partial \bar{x}^i}{\partial x^j} B^j \rightarrow \tilde{A}^k B^l = \frac{\partial \bar{x}^k}{\partial x^r} \frac{\partial \bar{x}^i}{\partial x^j} A^r B^j \rightarrow \tilde{C}^{kl} = \frac{\partial \bar{x}^k}{\partial x^r} \frac{\partial \bar{x}^i}{\partial x^j} C^{rj},
\]

where \( C^{rj} \) is called a 'contravariant' tensor of order '2'. Note that \( \tilde{C}^{kl} \) instantiates \( C^{rj} \). Moreover, there can be mixed tenors i.e.

\[
\tilde{R}^{pqrst}_{i} = \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial \bar{x}^r}{\partial x^s} \frac{\partial x^k}{\partial x^t} \frac{\partial \bar{x}^l}{\partial x^i} R^{qst}_{k}
\]

where \( R^{qst}_{k} \) is a fourth order tensor, contravariant in the indices 'q,s,t' and covariant in the index 'k'.
7.1 The Metric Tensor and Number Interference

Recall that if \( s(t) = x(t) + iy(t) \) and assuming the derivatives \( \dot{x}(t), \dot{y}(t) \) exist for all \( t \), then

\[
\frac{ds}{dt} = \frac{dx}{dt} \frac{dy}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \rightarrow ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt \rightarrow s
\]

The quantity \( \frac{ds}{dt} \) is instantiated by

\[
\frac{d\bar{s}}{dt} = \left| \begin{array}{cc} \frac{dx}{dt} \frac{dy}{dt} & \frac{dx}{dt} \frac{dy}{dt} \\ \frac{dy}{dt} & \frac{dx}{dt} \frac{dy}{dt} \end{array} \right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + 2 \left| \frac{dx}{dt} \right| \left| \frac{dy}{dt} \right|} \rightarrow \bar{s} = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + 2 \left| \frac{dx}{dt} \right| \left| \frac{dy}{dt} \right|} \, dt,
\]

which implies that \( \bar{s} \) instantiates \( s \).

Furthermore,

\[
\left(\frac{d\bar{s}}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + 2 \left| \frac{dx}{dt} \right| \left| \frac{dy}{dt} \right| \rightarrow d\bar{s}^2 = dx^2 + dy^2 + 2|dxdy|
\]

The term \( 2|dxdy| \) is a measure of the amount of number interference between the real and imaginary parts of a complex number in instantiation space. For example, let

\[ dx = dy = 1/2, \]

then

\[
\left(\frac{1}{2}, \frac{1}{2}\right) \rightarrow \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \left| \frac{1}{2} + \frac{1}{2} \right| = \left| \frac{1}{\sqrt{2}} \right| = 1
\]

This condition holds in general. To see this, given that \( x + y = 1 \), then \( x = 1 - y \).

Hence,

\[
x^2 + y^2 + 2xy = (1 - y)^2 + y^2 + 2(1 - y)y = 1 - 2y + y^2 + 2y - 2y^2 = 1
\]

However, this creates an instantiation anomaly. To see this, if \( x + y = 1 \), then

\[
s^2 = x^2 + y^2 = (1 - y)^2 = 1 - 2y + y^2 = 2y^2 - 2y + 1 \rightarrow s = \sqrt{2y^2 - 2y + 1}
\]
The locus of ‘s²’ is a parabola, which implies that the distance ‘s’ is a function of y. But if

\[ \forall y(s = 1), \]

then \( s(y) \rightarrow s = 1 \). Hence, more than one distance is instantiated by the distance ‘1’. Such instantiations are invalid. Moreover, the distance ‘s’ would be invariant if and only if the interference term in \( \bar{s} \) vanishes.

### 7.1.1 The Generalization of Number Interference

The instantiation ‘d²s’ can be generalized in the following manner:

\[ d\bar{s}^2 = g_{ij}dx^i dx^j \]

To see this, note that, in this case, it is immaterial that the tuples, such as ‘(x,y)’, are complex i.e.

\[ |\langle x,y \rangle| = \sqrt{x(x+y) + y(x+y)}, \]

which can be extended to three dimensions:

\[ |\langle x, y, z \rangle| = \sqrt{x(x+y+z) + y(x+y+z) + z(x+y+z)} \rightarrow |\langle x, y, z \rangle|^2 \]

In general

\[ |\langle x^1, x^2, \ldots, x^n \rangle| \]

\[ = \sqrt{x^1(x^1 + x^2 + \cdots + x^n) + x^2(x^1 + x^2 + \cdots + x^n) + \cdots + x^n(x^1 + x^2 + \cdots + x^n)} \]

\[ \rightarrow |\langle x^1, x^2, \ldots, x^n \rangle|^2 \]

\[ = x^1(x^1 + x^2 + \cdots + x^n) + x^2(x^1 + x^2 + \cdots + x^n) + \cdots + x^n(x^1 + x^2 + \cdots + x^n) \]

Replacing the \( x^i \)'s with ‘dx^i’s, then

\[ |\langle dx^1, dx^2, \ldots, dx^n \rangle|^2 \]

\[ = dx^1(dx^1 + dx^2 + \cdots + dx^n) + dx^2(dx^1 + dx^2 + \cdots + dx^n) + \cdots + dx^n(dx^1 + dx^2 + \cdots + dx^n) \]

To complete the generalization, introduce ‘\( g_{ij} \)', a matrix, which determines the extent of interference between \( x^i\)'s and the \( x^j\)'s, \( i \neq j \). Define

\[ d\bar{s}^2 = \sum_{i,j} g_{ij}dx^i dx^j \]

which represents a generalization of interference, assuming that the elements of \( g_{ij} \) are real numbers. For example, let
\[ g_{ij} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \rightarrow ds^2 = \sum_{i,j} g_{ij}dx^i dx^j = (dx^1)^2 + (dx^2)^2 + 2dx^1 dx^2, \quad i,j = 1,2 \]

If \( g_{ij} \) can be diagonalized, then the row vectors comprising \( g_{ij} \) are independent, signifying no interference. If any of the off-diagonal elements of the matrix ‘\( g_{ij} \)’ are non-zero, and, if there is no transformation which allows the diagonalization of \( g_{ij} \), then some level of interference exists between the \( x^i \)'s and \( x^j \)'s, \( i \neq j \). For example, let

\[ A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

By elementary matrix row operations,

\[ A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \rightarrow A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = B \]

Hence, the row vectors ‘\[ 1 \quad 1 \]’ and ‘\[ 0 \quad 1 \]’ are independent. Moreover, \( \det A = \det B = 1 \neq 0 \).

### 7.1.2 The Metric

However, in some situations, the interference terms vanish. For instance, let

\[ g_{ij} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \rightarrow d\bar{s}^2 = g_{ij}dx^i dx^j = (dx^1)^2 - dx^1 dx^2 + dx^2 dx^1 - (dx^2)^2 = (dx^1)^2 - (dx^2)^2, \quad i,j = 1,2 \]

In this case, the vectors that make up \( g_{ij} \) are dependent. In other words, the dimension of the matrix (the number of independent vectors) can be reduced. By ordinary matrix row operations,

\[ g_{ij} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \rightarrow d\bar{s}^2 = g_{ij}dx^i dx^j = (dx^1)^2 - dx^1 dx^2, \quad i,j = 1,2 \]

Once the matrix ‘\( g_{ij} \)’ is reduced by one dimension, an interference term reappears. In other words, \( g_{ij} \) has but one independent row vector. Note that \( \det g_{ij} = 0 \).

If \( g_{ij} \) is symmetric in the indices i.e. \( g_{ij} = g_{ji} \) and has an inverse ‘\( g^{ij} \)’ such that

\[ g_{ij}g^{kj} = \delta^k_i, \]

then \( g_{ij} \) is called a ‘metric’. In this case, the vectors that make up \( g_{ij} \) are independent. In general, if the \( g_{ij} \) is an \( n \times n \) matrix, where the \( 1 \times n \) row vectors that comprise the matrix are independent, then, by a well-known law of matrix theory, if the row vectors are independent, the matrix is diagonalizable. And if the matrix is diagonalizable, there is no interference between the \( x^i \)'s and the \( x^j \)'s, \( i \neq j \) within the formula for \( d\bar{s}^2 \).
7.1.2.1 When the Elements of the $g_{ij}$ are Functions of the Coordinates

More generally, if the elements of the matrix 'g$_{ij}$' are functions of the $x^i$'s, $g_{ij}$ becomes a metric if and only if the following conditions hold.

1. The functions have continuous first and second derivatives
2. $g_{ij} = g_{ji}$
3. $\det g_{ij} \neq 0$
4. $g_{ij}$ is a tensor

The first of these conditions ensures that the functions are sufficiently smooth so that the methods of the calculus can be applied. The second and third conditions ensure that $g_{ij}$ is ‘nonsingular’ (has an inverse). In other words, it guarantees that there is no interference between the $x^i$'s and the $x^j$'s. The fourth condition i.e.

$$
\bar{g}_{pq} d\bar{x}^p d\bar{x}^q = g_{jk} \frac{\partial x^j}{\partial \bar{x}^p} \frac{\partial x^k}{\partial \bar{x}^q} d\bar{x}^p d\bar{x}^q
$$

ensures that

$$
d\bar{s}^2 = g_{ij} dx^i dx^j
$$

is a scalar, and hence, an invariant, where $dx^i$ and $dx^j$ are considered contravariant vectors.

The importance of the metric tensor is due to its use in calculating 'arc-length'. For instance, suppose there is a curve ‘C’ specified in the coordinates

$$
x^i = x^i(t), \quad a \leq t \leq b \rightarrow \bar{s}(t) = \int_a^t \sqrt{\varepsilon g_{ij} \frac{dx^i}{du} \frac{dx^j}{du}} du,
$$

where $\bar{s}(t)$ is the 'arc-length' and

$$
\varepsilon = \begin{cases} 
1, & \text{if } g_{ij} \frac{dx^i}{du} \frac{dx^j}{du} \geq 0 \\
-1, & \text{if } g_{ij} \frac{dx^i}{du} \frac{dx^j}{du} < 0
\end{cases}
$$

For example, suppose

$$
g_{ij} = \begin{bmatrix} 
(x^1)^2 - 1 & 1 & 0 \\
1 & (x^2)^2 & 0 \\
0 & 0 & \frac{64}{9}
\end{bmatrix}, \quad \frac{64}{9}[(x^2)^2((x^1)^2 - 1) - 1] \neq 0
$$
Condition ’1’ is met because the functions that make up the elements of $g_{ij}$ are polynomials, and hence, their first and second derivatives exist. Condition ’2’ is met, since $g_{ij}$ is symmetric. The condition

$$\frac{64}{9} [(x^2)^2((x^1)^2 - 1) - 1] \neq 0$$

eNSURES that $g_{ij}$ is nonsingular. The reader can check to see that $g_{ij}$ is a tensor. Let a curve ‘$C$’ be parameterized as follows:

$$C = \begin{cases} 
x^1 = 2t - 1 \\
x^2 = 2t^2 \\
x^3 = t^3 
\end{cases}, \quad 0 \leq t \leq 1$$

Hence,

$$\epsilon \left( \frac{d\tilde{s}}{dt} \right)^2 = \left( \frac{dx^i}{dt} \right)^T g_{ij} \left( \frac{dx^j}{dt} \right) = \begin{bmatrix} 2 & 4t & 3t^2 \end{bmatrix} \begin{bmatrix} (2t - 1)^2 - 1 & 1 & 0 \\
1 & (2t^2)^2 & 0 \\
0 & 0 & \frac{64}{9} 
\end{bmatrix} \begin{bmatrix} 2t \\
3t^2 
\end{bmatrix}$$

$$= 64t^2 + 64t^4 + 16t^2 = (8t^3 - 4t)^2 \rightarrow \epsilon = 1$$

and

$$\tilde{s}(t) = \int_0^t (8u^4 - 4u) \, du = [2u^4 + 2u^2]_0^t = 2t^4 + 2t^2$$

Therefore, the length of the curve ‘$L$’ is

$$L = 2(1)^4 + 2(1)^2 = 4$$

If the same curve ‘$C$’ is specified in the coordinates

$$\tilde{x}^i = \bar{x}^i(t), \quad a \leq t \leq b \rightarrow s(t) = \int_a^t \frac{d\tilde{x}^i}{du} \frac{d\tilde{x}^j}{du} \, du \rightarrow s(t) = \tilde{s}(t)$$

7.1.3 The $g_{ij}$’s and Commutation

From a linear algebra standpoint, if the vectors that comprise $g_{ij}$ are independent, then the matrix can be diagonalized. On the other hand, if $g_{ij}$ cannot be diagonalized, then the vectors are dependent, which signifies number interference. Moreover, if some of the vectors that comprise $g_{ij}$ are dependent, it reduces the number of dimensions in the space. In addition, diagonalizable square matrices commute, which suggests that number interference is associated with matrices that do not commute.
7.2 Proper-Time

Recall that

\[
\frac{ds^2}{c^2} = d\tau^2 = \frac{1}{c^2} g_{\mu\nu} dx^\mu dx^\nu,
\]

where \(d\tau\) signifies the ‘proper-time’ element.

7.2.1 Dark- and Light- Proper-Time

Let

\[
0_{L\mu\nu}^\eta = \begin{bmatrix} c^2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad 0_{D\mu\nu}^\eta = \begin{bmatrix} k^2c^2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \rightarrow ds^2_L = 0_{L\mu\nu}^\eta dx^\mu dx^\nu,
\]

\[
ds^2_D = 0_{D\mu\nu}dx^\mu dx^\nu
\]

where \(0_{L\mu\nu}^\eta\) is the ‘Minkowski metric’ associated with light-light and \(0_{D\mu\nu}^\eta\) is the ‘Minkowski metric’ associated with dark-light. Hence,

\[
d\tau^2_L = \frac{1}{c^2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} dx^\mu dx^\nu, \quad d\tau^2_D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1/k^2 & 0 & 0 \\ 0 & 0 & -1/k^2 & 0 \\ 0 & 0 & 0 & -1/k^2 \end{bmatrix} dx^\mu dx^\nu,
\]

\[
d\tau^2_{D\rightarrow L} = \left(\frac{k^2 - 1}{k^2}\right) \sum_{i,j} g_{\mu\nu} dx^\mu dx^\nu, \quad g_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mu, \nu = 0, ..., 3
\]

Note that, in this case, the \(g_{\mu\nu}\) is not a metric. It is symmetric in the indices, but does not have an inverse.

Recalling that

\[
0_{L\mu\nu}^\eta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad 0_{D\mu\nu}^\eta = \begin{bmatrix} k^2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad c = 1,
\]

then the spatial dimensions are the same in both cases, but the time dimension is different. Hence,
\begin{align*}
(\partial_0 \eta_{\mu\nu} - \partial_\mu \eta_{\nu 0}) dx^\mu dx^\nu &= \sum_{lj} \begin{bmatrix}
    k^2 - 1 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
\end{bmatrix} dx^\mu dx^\nu \rightarrow (k^2 - 1) g_{\mu\nu}, \\
g_{\mu\nu} &= \begin{bmatrix}
    1 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1 \\
\end{bmatrix} \rightarrow ds^2_{D-L} = (k^2 - 1)c^2 \, dt^2 \rightarrow d\tau^2_{D-L} = (k^2 - 1) \, dt^2
\end{align*}

Note that
\begin{align*}
\begin{bmatrix}
    1 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 1 \\
\end{bmatrix} + \begin{bmatrix}
    0 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1 \\
\end{bmatrix} &= \begin{bmatrix}
    1 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1 \\
\end{bmatrix} = I \rightarrow (k^2 - 1) \begin{bmatrix}
    1 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
\end{bmatrix}
\end{align*}

and the $g_{\mu\nu}$ becomes a metric. It is symmetric in the indices and has an inverse.

Hence,
\begin{align*}
d\tau^2_{D-L} &= g_{\mu\nu} dx^\mu dx^\nu, \quad c = 1
\end{align*}

### 7.2.2 Dark- and Light- Generalized Proper-Time

In general, the proper-time for light- and dark-light respectively is given by
\begin{align*}
d\tau_L &= \frac{1}{c} \sqrt{g_{\mu\nu} dx^\mu dx^\nu}, \quad d\tau_D = \frac{1}{kc} \sqrt{g_{\mu\nu} dx^\mu dx^\nu}
\end{align*}

In this case, $g_{\mu\nu}$ is assumed to be a metric. Hence,
\begin{align*}
d\tau_L^2 &= g_{\mu\nu} dx^\mu dx^\nu, \\
k^2 d\tau_D^2 &= g_{\mu\nu} dx^\mu dx^\nu \rightarrow k^2 d\tau_D^2 - d\tau_L^2 = 0 \rightarrow d\tau_D = \frac{1}{k} d\tau_L \rightarrow \int d\tau_D = \frac{1}{k} \int d\tau_L \\
\tau_D &= \frac{\tau_L}{k}, \quad c = 1
\end{align*}

The dark proper-time lags behind the light-proper-time by a factor $'1/k \approx 10^{-1}'. $
7.3 Geodesics in Intrinsic Geometry

Given \(dl/dt = F(t, x, \dot{x})\), then

\[
I = \int_{t_1}^{t_2} F(t, x, \dot{x}) \, dt,
\]

where \(x\) is a function of \(t\). The condition that makes \(I\) an extremum is

\[
\frac{d}{dt} \left( \frac{\partial F}{\partial \dot{x}} \right) - \frac{\partial F}{\partial x} = 0.
\]

If \(F\) is a function of several variables i.e.

\[
F(t, x^1, \dot{x}^1, x^2, \dot{x}^2, x^3, \dot{x}^3, ..., x^n, \dot{x}^n),
\]

then the condition for an extremum becomes

\[
\frac{d}{dt} \left( \frac{\partial F}{\partial \dot{x}^k} \right) - \frac{\partial F}{\partial x^k} = 0, \quad k = 0, 1, 2, ..., n - 1,
\]

where there are \(n\) equations, one for each \(k\). The equations above represent a geodesic in intrinsic geometry. Importantly, the world line of a particle free from all external, non-gravitational forces, is a geodesic. The distance element '\(ds\)' is defined by

\[
ds = \sqrt{g_{\mu\nu} dx^\mu dx^\nu} \rightarrow \frac{ds}{c} = d\tau = \frac{1}{c} \sqrt{g_{\mu\nu} dx^\mu dx^\nu} \rightarrow \frac{dt}{dt} = \frac{1}{c} \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \rightarrow \tau
\]

\[
= \int \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \, dt, \quad c = 1,
\]

where \(\tau\) is the 'proper-time' and \(g_{\mu\nu}\) is the metric tensor. The proper-time is invariant and \(d\tau/dt\) represents the change in the proper-time given a small change in the coordinate time.

Image theory predicts two proper-times, 'dark-\(\tau_D\)' and 'light-\(\tau_L\)'

\[
d\tau_L = \frac{1}{c} \sqrt{g_{\mu\nu} dx^\mu dx^\nu},
\]

\[
d\tau_D = \frac{1}{c'} \sqrt{g_{\mu\nu} dx^\mu dx^\nu} = \frac{1}{kc} \sqrt{g_{\mu\nu} dx^\mu dx^\nu} \rightarrow d\tau_L = \sqrt{g_{\mu\nu} dx^\mu dx^\nu},
\]

\[
d\tau_D = \frac{1}{k} \sqrt{g_{\mu\nu} dx^\mu dx^\nu}, \quad c' = k, \quad c = 1
\]
Hence,

\[ \frac{d\tau_L}{dt} = \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}, \quad \frac{d\tau_D}{dt} = \frac{1}{k} \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \]

Let \( F_L = d\tau_L/dt \) and \( F_D = d\tau_D/dt \), then

\[ F_L = \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}, \quad F_D = \frac{1}{k} \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \rightarrow F_D - F_L = C \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}, \quad C = \frac{1 - k}{k} \]

Now

\[ \frac{\partial (F_D - F_L)}{\partial x^k} = \frac{1}{2} C (g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu)^{-1/2} \frac{\partial g_{\mu\nu}}{\partial x^k} \dot{x}^\mu \dot{x}^\nu, \quad \frac{\partial (F_D - F_L)}{\partial \dot{x}^k} = C (g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu)^{-1/2} g_{\mu k} \dot{x}^\mu \]

Hence,

\[ \frac{d}{dt} \left( \frac{\partial (F_D - F_L)}{\partial \dot{x}^k} \right) - \frac{\partial (F_D - F_L)}{\partial \dot{x}^k} = \frac{d}{dt} \left( C (g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu)^{-1/2} g_{\mu k} \dot{x}^\mu \right) - \frac{1}{2} C (g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu)^{-1/2} \frac{\partial g_{\mu\nu}}{\partial x^k} \dot{x}^\mu \dot{x}^\nu = 0 \]

\[ \rightarrow \frac{d}{dt} \left( (g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu)^{-1/2} g_{\mu k} \dot{x}^\mu \right) - \frac{1}{2} (g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu)^{-1/2} \frac{\partial g_{\mu\nu}}{\partial x^k} \dot{x}^\mu \dot{x}^\nu = 0 \]

Let \( \dot{\tau} = \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \), then

\[ \frac{d}{dt} \left( \frac{\partial (F_D - F_L)}{\partial \dot{x}^k} \right) = \frac{d}{dt} \left( (g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu)^{-1/2} g_{\mu k} \dot{x}^\mu \right) = \frac{d}{dt} \left( \frac{g_{\mu k} \dot{x}^\mu}{\dot{\tau}} \right) = \frac{\left( g_{\mu k} \dot{x}^\mu + \frac{\partial g_{\mu k} \dot{x}^\mu}{\partial x^v} \dot{x}^v \right)}{\dot{\tau}^2} - \frac{\dot{\tau} g_{\mu k} \dot{x}^\mu}{\dot{\tau}^2} \]

Hence,

\[ \frac{d}{dt} \left( \frac{\partial (F_D - F_L)}{\partial \dot{x}^k} \right) - \frac{\partial (F_D - F_L)}{\partial \dot{x}^k} = \left( g_{\mu k} \dot{x}^\mu + \frac{\partial g_{\mu k} \dot{x}^\mu}{\partial x^v} \dot{x}^v \right) \frac{\dot{\tau}}{\dot{\tau}^2} = \frac{\dot{\tau} g_{\mu k} \dot{x}^\mu}{\dot{\tau}^2} - \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^k} \dot{x}^\mu \dot{x}^\nu = 0 \]

Note that

\[ \frac{\partial g_{\mu k} \dot{x}^\mu \dot{x}^v}{\partial x^v} = \frac{1}{2} \left( \frac{\partial g_{\mu k}}{\partial x^v} + \frac{\partial g_{vk}}{\partial x^k} \right) \dot{x}^\mu \dot{x}^v \rightarrow g_{\mu k} \dot{x}^\mu + \frac{1}{2} \left( \frac{\partial g_{\mu k}}{\partial x^v} + \frac{\partial g_{vk}}{\partial x^k} \right) \dot{x}^\mu \dot{x}^v - \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^k} \dot{x}^\mu \dot{x}^v \]

\[ = g_{\mu k} \dot{x}^\mu + [\mu, k] \dot{x}^\mu \dot{x}^v = \frac{\dot{\tau} g_{\mu k} \dot{x}^\mu}{\dot{\tau}}, \]

159
where \([\mu\nu, k]\) is the Christoffel symbol of the first kind. If \(\tau\) is chosen as a parameter, then \(\dot{\tau} = 1\), \(\ddot{\tau} = 0\). Hence,

\[
g_{\mu k} \dddot{x}^\mu + [\mu\nu, k] \dot{x}^\mu \dddot{x}^\nu = g_{\mu k} \frac{d^2 x^\mu}{d\tau^2} + [\mu\nu, k] \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0
\]

Multiplying through by \(g^{rk}\) gives

\[
\frac{d^2 x^r}{d\tau^2} + \Gamma^r_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0,
\]

which is the equation of a geodesic. Since light- and dark-proper-times are extremums, both are geodesics, which implies that the difference \(F_D - F_L\) is a geodesic. In other words, since the dark- and light-proper-times differ by a constant, the stationarity of the difference between them remains.

### 7.3.1 The Covariant Derivative of the Tangent Vector

If a space curve \('C'\) is immersed in a vector field, computing how a scalar function \('\varphi'\) varies over \(C\) is accomplished by choosing a parameter that varies along the curve, such as the proper-time \('\tau'\). The change in \(\varphi\) with respect to \(\tau\) is given by the chain rule:

\[
\frac{d\varphi}{d\tau} = \frac{\partial \varphi}{\partial y^m} \frac{dy^m}{d\tau},
\]

where \(\frac{dy^m}{d\tau}\) is the unit tangent vector along the curve i.e. \(|\frac{dy^m}{d\tau}| = 1\). If \(\varphi\) is replaced by a vector \('V^n'\), then

\[
\frac{dV^n}{d\tau} = \frac{\partial V^n}{\partial y^m} \frac{dy^m}{d\tau}
\]

Since \(\frac{\partial V^n}{\partial y^m}\) is the ordinary derivative of a vector, it is not, in general, a tensor. To make \(\frac{\partial V^n}{\partial y^m}\) a tensor, it must be replaced by the covariant derivative. Let

\[
\nabla_\tau V^n(y) = \nabla_m V^n(y) \frac{dy^m}{d\tau} \rightarrow \nabla_\tau V^n(y) = \left(\frac{\partial V^n}{\partial y^m} + \Gamma^m_{mr} V^r\right) \frac{dy^m}{d\tau} = \frac{\partial V^n}{\partial y^m} \frac{dy^m}{d\tau} + \Gamma^m_{mr} V^r \frac{dy^m}{d\tau}
\]

The equation on the right above is the covariant derivative of a vector, which is a tensor and describes how the vector varies along \(C\). To determine how the tangent vector varies along \(C\), simply substitute the tangent vector for \(V^n\) i.e. \(V^n \rightarrow d y^n / d\tau\), then

\[
\left(\frac{dV^n}{d\tau} + \Gamma^m_{mr} V^r\right) \frac{dy^m}{d\tau} \rightarrow \frac{d^2 y^n}{d\tau^2} + \Gamma^m_{mr} \frac{dy^r}{d\tau} \frac{dy^m}{d\tau},
\]

which is the covariant derivative of the tangent vector. Since the covariant derivative is a tensor, if it is zero in one reference frame, it is zero in all reference frames. The equation above becomes the equation of a geodesic if it is set equal to zero.
7.4 Curvature and the Parallel Transport of a Vector

Determining whether a space is geometrically curved or flat requires the concept of the ‘parallel transport’ of a vector along a curve. Mathematically, the parallel transport of an arbitrary vector $V^n$ along a curve is defined as

$$\nabla_s V^n(y) = \frac{dV^n}{ds} + \Gamma^m_{mr} V^r \frac{dy^m}{ds} = 0 \rightarrow dV^n = -\Gamma^m_{mr} V^r dy^m$$

Note that the change in the vector $V^n$ depends on the Christoffel symbols. In other words, if the covariant derivative of a vector is equal to zero along the curve, then the vector is ‘parallel transported’. If $V^n$ is the tangent vector, then the curve is called a ‘geodesic’. A geodesic is a curve whose curvature is entirely due to the intrinsic geometry of the space. If a curve has curvature of its own, independent of the curvature of the space, it is not a geodesic.

7.4.1 Curved vs. Flat Spaces

If an arbitrary vector is parallel transported along a closed curve from a point $P$, and upon returning to $P$, is the same vector, the geometry enclosed by the curve is ‘flat’. Otherwise, the geometry of the enclosed space has ‘curvature’.

In a curved space, a small angle deficit $\delta\theta$ is proportional to the area bounded by a closed curve i.e. $\delta\theta = R\delta A$, where $R$ is proportional to the ‘curvature’ and $\delta A$ is the area bounded by the curve. If a point on a surface is circumvented by a closed curve, the curve will surround a small area of the surface. If an arbitrary vector is parallel transported along the curve, then $d\theta/dA = R$, where $d\theta$ is the change in angle the vector makes with itself upon its return, $dA$ is the area enclosed by the curve and $R$ is proportional to the curvature around the point. If $d\theta/dA = R = 0$, there is no curvature around the point and the space enclosed by the curve is flat.

Another way of looking at curvature is to examine the Riemann-Christoffel tensor:

$$R^\alpha_{\mu\nu\beta} = \frac{\partial \Gamma^\alpha_{\mu\beta}}{\partial x^\nu} - \frac{\partial \Gamma^\alpha_{\nu\beta}}{\partial x^\mu} + \Gamma^\alpha_{\nu\delta} \Gamma^\delta_{\mu\beta} - \Gamma^\alpha_{\nu\beta} \Gamma^\delta_{\mu\delta}$$

Note that the Riemann-Christoffel tensor is made up of the Christoffel symbols. If the Christoffel symbols vanish, then

$$R^\alpha_{\mu\nu\beta} = 0$$

A $g_{ij}$ specified in a coordinate system $'x^i'$ is called a ‘Euclidean metric’ if, under some permissible coordinate transformation, $\bar{g}_{ij} = \delta_{ij}$. Recall that

$$\Gamma^r_{pm} = \frac{1}{2} g^{nr} \left( \frac{\partial g_{mn}}{\partial y^p} + \frac{\partial g_{pn}}{\partial y^m} - \frac{\partial g_{mp}}{\partial y^n} \right)$$
However, if the $g_{ij}'s$ are replaced by the $\delta_{ij}'s$ within $\Gamma_{pm}$, then the expression within the curly brackets vanishes, since the derivative of a constant is zero. Hence, all the Christoffel symbols within $R^\alpha_{\mu\nu\beta}$ vanish. But since $R^\alpha_{\mu\nu\beta}$ is a tensor, if it is zero in one set of coordinates, it is zero in all sets of permissible coordinates. Therefore, a space is called ‘Euclidean flat’ if and only if

$$R^g_{\mu\nu\beta} = 0$$

### 7.4.2 Computing Curvature

To compute the amount curvature, two pieces of information are required: 1) the orientation of the plane in which the curve sits, 2) the change in angle the vector makes with itself when parallel transported along the curve. If the coordinates $'x^\mu'$ and $'x'^\nu$ define the plane in which a point on a surface or hypersurface is transported, then $dA \propto dx^\mu dx^\nu$. The area will not exactly equal $dx^\mu dx^\nu$, since the coordinate axes are not necessarily perpendicular. If an arbitrary vector $'V'^\alpha$ is parallel transported counterclockwise around a parallelogram and if there is intrinsic curvature in the space, the change in the direction of the vector will be proportional to the area enclosed by the parallelogram i.e.

$$dV^\alpha \propto dx^\mu dx^\nu V^\tau$$

Changing $'\propto'$ to '$=$' requires introducing a factor $'R^\alpha_{\sigma\nu\tau}'$ such that

$$dV^\alpha = R^\alpha_{\mu\nu\tau} dx^\mu dx^\nu V^\tau$$

### 7.4.3 The Curvature Tensor

The factor $'R^\alpha_{\mu\nu\tau}'$ is called the ‘curvature tensor’. It signifies the change in the vector $'V'^\alpha$, given that the vector is parallel transported around a closed loop. Recall the definition of parallel transport i.e.

$$dV^\alpha = -\Gamma^\alpha_{\sigma\tau} V^\tau dx^\sigma$$

If $dV^\alpha$ depends on the $\Gamma^\alpha_{\sigma\tau}'s$ (Cristoffel symbols), then the change in the vector depends on the derivatives of the $\Gamma^\alpha_{\sigma\tau}'s$, which is made up of the metric tensor and its derivatives. The tensor $'R^\mu_{\sigma\nu\tau}'$ then must be constructed from the Christoffel symbols. The tensor that satisfies this requirement is the ‘Riemann-Christoffel tensor’ (see Book II: Sec. 13.12). Within $R^\alpha_{\mu\nu\tau}$, the two indices $'\mu, \nu'$ define the plane of transportation while the indices $'\alpha, \tau'$ define the amount of angle variation as the vector is parallel transported around the curve.

### 7.4.3.1 Commutation and Curvature

The commutator $'[A, B]'$ is defined

$$[A, B] = AB - BA$$
If $A, B$ are simply numbers, then $[A, B] = 0$. But consider

$$
\left[\frac{\partial}{\partial x}, f(x)\right] = \frac{\partial f}{\partial x} - f \frac{\partial}{\partial x} \frac{\partial f}{\partial x} V - f \frac{\partial V}{\partial x},
$$

The ‘$V$’ arises because $\partial / \partial x$ must act on something, in this case, that ‘something’ is $V$. The first term of the expression on the right above is the differentiation of a product. Hence,

$$
\frac{\partial}{\partial x} (fV) - f \frac{\partial V}{\partial x} = \frac{\partial f}{\partial x} V + f \frac{\partial V}{\partial x} - f \frac{\partial V}{\partial x} \rightarrow \left[\frac{\partial}{\partial x}, f(x)\right] = \frac{\partial f}{\partial x}
$$

Suppose an arbitrary vector is parallel transported around the loop ‘$OBCDO’’ (see fig. 7.4.3.1-1). Note that $O, O'$ is the same point, labeled differently. Consider

$$(V_C - V_D) - (V_B - V_O)$$

$$(V_C - V_B) - (V_D - V_O')$$

The expression ‘$(V_C - V_D)$’ represents the difference in the vector ‘$V$’ at positions ‘$C$’ and ‘$D$’. The difference ‘$(V_C - V_D) - (V_B - V_O)$’ represents the change in the vector ‘$V$’ along the $x^\nu$-direction. Likewise, ‘$(V_C - V_B) - (V_D - V_O')$’ represents the change of the vector along the $x^\mu$-direction. As a result,

$$(V_C - V_D) - (V_B - V_O) - [(V_C - V_B) - (V_D - V_O')] = V_O - V_O' = dV,$

where $dV$ is the total change in the vector ‘$V$’ as it is parallel transported around the loop ‘$OBCDO’’.

Computing the change in a vector along a given direction requires taking the gradient i.e.

$$V_C - V_D = \frac{\partial V^\alpha}{\partial x^\nu} dx^\nu \rightarrow V_C - V_D = \nabla_\nu V^\alpha dx^\nu, \quad (V_B - V_O) = \nabla_\nu V^\alpha dx^\nu$$

where the ordinary derivative ‘$\partial V / \partial x^\nu$’ has been replaced by the covariant derivative ‘$\nabla_\nu$’. Likewise,
\[ V_C - V_B = \frac{\partial V^\alpha}{\partial x^\mu} dx^\mu \rightarrow V_C - V_B = \nabla_\mu V^\alpha dx^\mu, \quad (V_D - V_O) = \nabla_\mu V^\alpha dx^\mu \]

Hence,

\[
(V_C - V_D) - (V_B - V_D) - [(V_C - V_B) - (V_D - V_O)] = V_O - V_O'
\]

\[
= \nabla_\nu \nabla_\mu V^\alpha dx^\mu dx^\nu - \nabla_\mu \nabla_\nu V^\alpha dx^\mu dx^\nu = (\nabla_\nu \nabla_\mu - \nabla_\mu \nabla_\nu) V^\alpha dx^\mu dx^\nu = dV^\alpha
\]

Note that

\[
\nabla_\nu \nabla_\mu - \nabla_\mu \nabla_\nu = [\nabla_\nu, \nabla_\mu]
\]

and if \([\nabla_\nu, \nabla_\mu] = 0\), then \(dV^\alpha = 0\), indicating the space is flat. If \([\nabla_\nu, \nabla_\mu] \neq 0\), the space has curvature.

Recall that the covariant derivative takes the form

\[
\nabla_\nu = \frac{\partial}{\partial x^\nu} + \Gamma^\nu_{\nu\nu}
\]

Hence,

\[
[\nabla_\nu, \nabla_\mu] = \left(\frac{\partial}{\partial x^\nu} + \Gamma_\nu\right)\left(\frac{\partial}{\partial x^\mu} + \Gamma_\mu\right) - \left(\frac{\partial}{\partial x^\mu} + \Gamma_\mu\right)\left(\frac{\partial}{\partial x^\nu} + \Gamma_\nu\right) =
\]

\[
= \frac{\partial}{\partial x^\nu} \frac{\partial}{\partial x^\mu} + \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} + \Gamma_\nu \frac{\partial}{\partial x^\mu} + \Gamma_\mu \frac{\partial}{\partial x^\nu} - \left(\frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} + \frac{\partial}{\partial x^\nu} \frac{\partial}{\partial x^\mu} + \Gamma_\nu \Gamma_\mu - \Gamma_\mu \Gamma_\nu\right)
\]

\[
= - \left[ \frac{\partial}{\partial x^\mu}, \Gamma_\nu \right] + \left[ \frac{\partial}{\partial x^\nu}, \Gamma_\mu \right] + \left[ \Gamma_\nu, \Gamma_\mu \right]
\]

where, for brevity, \(\Gamma_\nu\) represents the Christoffel symbol corresponding to the \(\nu\)-direction and so on.

Remembering that

\[
\left[ \frac{\partial}{\partial x^\nu}, f(x) \right] = \frac{\partial f}{\partial x} \rightarrow \left[ \frac{\partial}{\partial x^\mu}, \Gamma_\nu \right] = \frac{\partial \Gamma_\nu}{\partial x^\mu}, \quad \left[ \frac{\partial}{\partial x^\nu}, \Gamma_\mu \right] = \frac{\partial \Gamma_\mu}{\partial x^\nu}
\]

and that the commutator \(\left[ \nabla_\nu, \nabla_\mu \right]\) is made up of Christoffel symbols and their derivatives, it then follows that \([\nabla_\nu, \nabla_\mu]_\beta = R^\alpha_{\mu\nu\beta}\) is a tensor, since the outer multiplication of two tensors is a tensor and the subtraction of two tensors of the same rank is a tensor. Hence,

\[
dV^\alpha = \left(\frac{\partial \Gamma^\alpha_{\beta\mu}}{\partial x^\nu} - \frac{\partial \Gamma^\alpha_{\nu\beta}}{\partial x^\mu} + \Gamma^\alpha_\delta \Gamma^\delta_{\mu\beta} - \Gamma^\alpha_\nu \Gamma^\nu_{\delta\mu\beta}\right) dx^\mu dx^\nu V^\beta = R^\alpha_{\mu\nu\beta} dx^\mu dx^\nu V^\beta
\]

is the change in the vector \(dV^\alpha\) around a closed loop. Note that \(dV^\alpha\) is a contravariant vector.
7.4.3.2 Curvature and the Ricci Tensor

If \( R^\alpha_{\mu \nu \beta} \) is contracted on the indices \( '\alpha, \nu' \), then

\[
R^\alpha_{\mu \nu \beta} \rightarrow R^\alpha_{\mu \alpha \beta} = R_{\mu \beta}
\]

The tensor \( 'R_{\mu \beta}' \) is called the ‘Ricci tensor’ (contraction of a tensor is a tensor). In particular, the tensor \( 'R_{\mu \beta}' \) can be used to compute the change in the direction of a vector:

\[
dV = R_{\mu \beta} d\mu \nu \nu
\]

Note that \( dV \) is a scalar, and hence, invariant. Therefore, the change in the vector is independent of the coordinate system used to describe the parallel transport of the vector.

7.4.3.3 The Curvature Scalar

While \( R^\alpha_{\mu \nu \beta} \) is not, \( R_{\mu \beta} \) is a symmetric tensor. Furthermore, \( R_{\mu \beta} \) can be contracted i.e. \( g^{\mu \beta} R_{\mu \beta} = R \), which is called the ‘curvature scalar’. If \( R \neq 0 \), the space is curved. Interestingly, the fact that \( R = 0 \) does not guarantee a Euclidean flat space. The condition \( 'R = 0' \) is a necessary, but not a sufficient condition for Euclidean flatness. If the space has one or two dimensions and \( R = 0 \), then the space is Euclidean flat. But the rule does not extend to 3- or higher-dimensional spaces [273].

7.5 Einstein’s Search for a Gravitational Tensor Equation

To complete his theory of gravity, Einstein had to find a tensor \( 'G_{\mu \nu}' \) such that

\[
G_{\mu \nu} = -8\pi G T_{\mu \nu},
\]

where \( T_{\mu \nu} \) is the ‘energy-momentum’ tensor and, since energy and momentum taken together, are locally conserved, \( \nabla_{\mu} T_{\mu \nu} = 0.2 \)

The tensor \( 'G_{\mu \nu}' \) had to be of rank \( '2' \) and contain the derivatives of \( g_{\mu \nu} \)’s. Einstein first tried the Ricci tensor:

\[
R_{\mu \nu} = -8\pi G T_{\mu \nu}, \quad G_{\mu \nu} = R_{\mu \nu}
\]

He immediately ran into difficulties, since \( \nabla_{\mu} T_{\mu \nu} = 0 \), but, in general, \( \nabla_{\mu} R_{\mu \nu} \neq 0 \). Einstein wanted energy and momentum conservation to follow as a consequence of the curvature of space-time. He did not achieve this. However, the only combination of the Ricci tensor, the scalar of curvature \( 'R' \) and the metric tensor that has this property is

\[\]
\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \]

Working out the calculation for

\[ \nabla_{\mu} \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = 0 \]

gives

\[ \nabla_{\mu} R_{\mu\nu} = \frac{1}{2} \nabla_{\mu} (g_{\mu\nu} R) = \frac{1}{2} \nabla_{\mu} g_{\mu\nu} R + \frac{1}{2} g_{\mu\nu} \nabla_{\mu} R = \frac{1}{2} g_{\mu\nu} \nabla_{\mu} R, \]

since \( \nabla_{\mu} g_{\mu\nu} = 0 \). Hence,

\[ \nabla_{\mu} \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = 0 \rightarrow \nabla_{\mu} \left[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right] = -8\pi G \nabla_{\mu} T_{\mu\nu} \]

and this implies that

\[ G_{\mu\nu} = -8\pi G T_{\mu\nu}, \quad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \]

These are the equations for the gravitational field.

### 7.5.1 The Cosmological Constant

One more term can be added to Einstein’s equations. This arises because \( \nabla_{\mu} g_{\mu\nu} = 0 \). Hence,

\[ G_{\mu\nu} = -8\pi G T_{\mu\nu} \rightarrow G_{\mu\nu} + \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu}, \]

where \( \Lambda \) is called the ‘cosmological constant’. It represents a force which counteracts the gravitational force and is proportional to distance ‘\( r \)’. The value of \( \Lambda \) is tiny. It plays virtually no role in the gravitational equations when \( r \) is small, but only when \( r \) is extremely large, in fact, cosmologically large. The cosmological constant is included in Einstein’s equations to account for the expansion of the Universe.

### 7.6 Einstein’s Equations Due to Intrinsic Time

In general relativity, the element of proper-time ‘\( \tau \)’ is

\[ d\tau = \frac{1}{c} \sqrt{g_{\mu\nu} dx^\mu dx^\nu} \rightarrow d\dot{t} = \frac{1}{c} \sqrt{g_{\mu\nu} d\dot{x}^\mu d\dot{x}^\nu} \rightarrow \tau = \int_{t_1}^{t_2} \frac{1}{c} \sqrt{g_{\mu\nu} d\dot{x}^\mu d\dot{x}^\nu} \, dt \]

The extremum of the equation on the right above is given by the equation of a geodesic:

\[ \frac{d^2 x^r}{d\tau^2} + \gamma_{pq}^r \frac{dx^p}{d\tau} \frac{dx^q}{d\tau} = 0 \]
Assuming that the Christoffel symbols vanish (the space is flat), then the equation above reduces to

\[ \frac{d^2x^r}{d\tau^2} = 0, \]

which is just an ordinary derivative. If

\[ \frac{dx^1}{d\tau} = \frac{dx^2}{d\tau} = \frac{dx^3}{d\tau} = 0, \]

signifying no motion in the spatial dimensions, then

\[ \frac{dx^0}{d\tau} = 1 \rightarrow d\tau = dt \]

and the proper-time is equal to the coordinate time.

**7.6.1 Intrinsic Time**

But, according to image theory, an observer will find space expanding on account of an intrinsic time \( \tau \) expressed

\[ d\tau = \frac{dt}{\sqrt{1 - \left( \frac{\bar{v}(t)}{c} \right)^2}}, \quad \bar{v}(t) = Gt, \quad G = \frac{e^{10^{15}}}{c^2}, \]

where there is a hypothesized constant acceleration in the direction of the coordinate time. Hence,

\[ \frac{d\tau}{dt} = \frac{1}{\sqrt{1 - \frac{G^2t^2}{c^2}}}, \quad \frac{d}{dt}\left( \frac{d\tau}{dt} \right) = \frac{d^2\tau}{dt^2} = \frac{G^2t}{c^2 \left( \frac{1}{c^2} - \frac{G^2t^2}{c^2} \right)^{3/2}}. \]

However,

\[ \frac{G^2t}{c^2 \left( \frac{1}{c^2} - \frac{G^2t^2}{c^2} \right)^{3/2}} = 0 \rightarrow t = 0 \]

In other words, if \( t = 0 \), there is no ‘intrinsic time’. 
Suppose that
\[
\sqrt{1 - \frac{[l]^2 G^2 t^2}{c^2}} = \sqrt{\eta_{\mu\nu} dx^\mu dx^\nu} \to d\tau^2 = \gamma^2 \eta_{\mu\nu} dx^\mu dx^\nu = \gamma^2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} dx^\mu dx^\nu, 
\]
\[c = 1, \quad \gamma^2 = \frac{1}{1 - G^2 t^2}\]

If \( t = 0 \), then
\[\nabla_\mu \gamma^2 \eta_{\mu\nu} = 0\]

Evidently, in a universe devoid of matter and energy, where \( t = 0 \) (no intrinsic time), all objects move along geodesics. However, in a universe where there is intrinsic time i.e. \( t \) varies, objects no longer move along geodesics, since \( \nabla_\mu \lambda^2 \eta_{\mu\nu} \neq 0 \). There must be a force causing a deviation from the geodesic. According to image theory, Einstein’s equations are only valid in a universe in which there is no ‘intrinsic time’.

### 7.6.2 Einstein’s Equations Modified

If there is such a thing as ‘intrinsic time’, Einstein’s field equations would need modification. Recall that
\[\nabla_\mu \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = 0\]

If there is no deformation of the space, but only a change in volume or area, then
\[R_{\mu\nu} \approx \lambda g_{\mu\nu}, \quad R \approx n\lambda,\]
where \( n \) is equal to the number of dimensions. For details of how this comes about consult [273]. Let
\[\lambda = \tan^{-1} \left( \frac{Gt}{\sqrt{1 - G^2 t^2}} \right) \to \frac{d\lambda}{dt} = \frac{1}{\sqrt{1 - G^2 t^2}} \implies \left( \frac{d\lambda}{dt} \right)^2 = \frac{1}{1 - G^2 t^2}, \quad c = 1\]

Hence, let
\[d\tau^2 = \gamma^2 g_{\mu\nu} dx^\mu dx^\nu \to d\tau = \gamma \sqrt{g_{\mu\nu} dx^\mu dx^\nu}, \quad \gamma = \frac{d\lambda}{dt} = \frac{1}{\sqrt{1 - G^2 t^2}},\]

where \( \tau \) is now called the ‘intrinsic proper-time’.

If
\[R_{\mu\nu} \approx \gamma^2 g_{\mu\nu}, \quad R \approx n\gamma^2,\]
then
\[ \nabla_{\mu}[\gamma^2 g_{\mu\nu} - 2 g_{\mu\nu} \gamma^2] = -\nabla_{\mu}(\gamma^2 g_{\mu\nu}) = -(\nabla_{\mu}\gamma^2 g_{\mu\nu} + \gamma^2 \nabla_{\mu} g_{\mu\nu}) = -\nabla_{\mu} \gamma^2 g_{\mu\nu}, \quad n = 4, \]

since \( \nabla_{\mu} g_{\mu\nu} = 0 \). However, \( \nabla_{\mu} \gamma^2 g_{\mu\nu} \neq 0 \). Evidently, the only scenario in which
\[ \nabla_{\mu} \left( \gamma^2 g_{\mu\nu} - \frac{1}{2} g_{\mu\nu} n \gamma^2 \right) = 0 \to n = 2 \]

The analysis above only applies to a specific case, where \( n = 2 \). The solution requires a tensor equation that would hold in general. By hypothesis, let
\[ \frac{1}{\gamma^2} R_{\mu\nu} = -8\pi \sqrt{G} T_{\mu\nu} \to R_{\mu\nu} = -8\pi \sqrt{G} \gamma^2 T_{\mu\nu}, \quad c = 1, \]

where \( R_{\mu\nu} \) is the Ricci tensor, which is similar to Einstein’s original hypothesis. Now
\[ \nabla_{\mu} R_{\mu\nu} = -8\pi \sqrt{G} \nabla_{\mu} (\gamma^2 T_{\mu\nu}) = -8\pi \sqrt{G} (\nabla_{\mu} \gamma^2 T_{\mu\nu} + \gamma^2 \nabla_{\mu} T_{\mu\nu}) = -8\pi \sqrt{G} \nabla_{\mu} \gamma^2 T_{\mu\nu}, \]

since \( \nabla_{\mu} T_{\mu\nu} = 0 \). And this implies
\[ R_{\mu\nu} = -8\pi \sqrt{G} \gamma^2 T_{\mu\nu}, \quad \gamma^2 = \frac{1}{1 - G^2 t^2}, \quad c = 1 \]

Is \( \gamma \) a tensor? Note that
\[ \lambda = \tan^{-1} \left( \frac{G t}{\sqrt{1 - G^2 t^2}} \right) \]

is simply an arbitrary function of time. Hence, \( \gamma = d\lambda / dt \) is the derivative of an arbitrary function, which can be written as the product of a contravariant and a covariant vector. Specifically, the gradient of an arbitrary function ‘\( \lambda \)’ is a covariant vector i.e. \( T_i = \partial \lambda / \partial x^i \) and the tangent vector is a contravariant vector i.e. \( S^i = dx^i / dt \). Hence,
\[ T_i S^i = \frac{\partial \lambda}{\partial x^r} \frac{dx^r}{dt} = \frac{d\lambda}{dt} = \gamma \]

So evidently,
\[ (T_i S^i)^2 = \gamma^2 \]

is a scalar, and hence, invariant.

To see that this is compatible with conservation of momentum and energy, let
\[ \nabla_{\mu} R_{\mu\nu} = -8\pi \sqrt{G} \nabla_{\mu} (\gamma^2 T_{\mu\nu}) = -8\pi \sqrt{G} (\nabla_{\mu} \gamma^2 T_{\mu\nu} + \gamma^2 \nabla_{\mu} T_{\mu\nu}) \to \nabla_{\mu} R_{\mu\nu} + 8\pi \sqrt{G} \nabla_{\mu} \gamma^2 T_{\mu\nu} = -8\pi \sqrt{G} \gamma^2 \nabla_{\mu} T_{\mu\nu} \to R_{\mu\nu} = -8\pi \sqrt{G} \gamma^2 T_{\mu\nu}, \quad \nabla_{\mu} T_{\mu\nu} = 0 \]
Note that if $T_{\mu\nu} = 0$, representing a universe devoid of matter and energy, the space would be ‘Ricci flat’ (no intrinsic time). Moreover, there is curvature in space-time if $T_{\mu\nu} \neq 0$, which implies that $\nabla_{\mu} R_{\mu\nu} \neq 0$. The change in curvature takes place only in the time dimension, since

$$\nabla_{i} R_{ij} = 0, \quad i = 1, ..., 3,$$

because the $\nabla_{i} (\gamma^2)$ in the spatial dimensions vanish, since $\gamma^2$ is strictly a function of intrinsic time.

### 7.6.3 Elimination of the Cosmological Constant

Evidently then, there would be no need to add a cosmological constant to the gravitational equations. This is not as far-fetched as it might appear. Note that the value of cosmological constant ‘$\Lambda$’ is

$$\Lambda \approx 1.2 \times 10^{-50} \text{ [cm}^{-2}]$$

as compared to

$$\frac{[l^2]G^2}{c^2} \approx 8.21 \times 10^{-51} \text{ [sec}^{-2}]$$

Each of these constants is associated with spatial expansion.

### 7.6.4 Conjecture on the Physics at Small-Scales

The general theory of relativity provides the physics at the large-scale. As a matter of conjecture, suppose that $R_{\mu\nu}$ and $T_{\mu\nu}$ have inverses ‘$S^{\mu\nu}$’ and ‘$U^{\mu\nu}$’ respectfully. Could ‘$S^{\mu\nu}$’ and ‘$U^{\mu\nu}$’ be part of a tensor equation representing the physics at the small-scale? Of course, both $R_{\mu\nu}$ and $T_{\mu\nu}$ are not generally invertible. Being such would forbid a universe devoid of matter and energy. However, mathematically, the conjecture is feasible. To see this, let

$$R_{\mu\nu} = -8\pi \sqrt{G} \gamma^2 T_{\mu\nu},$$

then

$$R_{\mu\nu} g_{\nu i} = -8\pi \sqrt{G} \gamma^2 T_{\mu\nu} g_{\nu i} \rightarrow - \frac{R_{\mu}}{8 \pi \sqrt{G} \gamma^2} T_{\mu i} \rightarrow - \frac{R_{\mu} g_{\mu j}}{8 \pi \sqrt{G} \gamma^2} = T_{\mu i} g_{\mu j} \rightarrow U^{\mu\nu} = - \frac{S^{\mu\nu}}{8 \pi \sqrt{G} \gamma^2}$$

$$\rightarrow R_{1v} S^{j v} = \delta_{i}^{j}$$

The conjecture is presented here in preparation for chapter 9, which discusses unifying quantum physics with relativity.
7.7 How to Build a Black Hole

Consider the diagram in fig. 7.7-1, where two masses are surrounded by an electric current directed into the plane of page.

\[ \sqrt{f_t} = -\sqrt{G m_1} \]

\[ \sqrt{f_s} = i \frac{\sqrt{G m_1 m_2}}{r^2} \]

\[ \sqrt{f_t} = \sqrt{G m_2} \]

Figure 7.7-1

If, as image theory claims, there is an acceleration in the time dimension proportional to \( G \), then let there be two masses \( m_1, m_2 \) a distance \( r \) apart (see fig. 7.7-1). If \( m_1, m_2, r = 1 \), then

\[ \sqrt{f_t} = -\sqrt{G m_1} \frac{d\tau}{dt}, \quad G = \frac{e}{10^{15}}, \quad m_1 = 1 \]

As the two masses begin moving relative to one another on account of the force \( G \) in the direction of time, a stationary observer at \( x \) would witness, over time, an increase in the masses proportional the intrinsic time. Evidently, the momentum \( p_\tau \) is given by

\[ f_\tau \, dt = G m_1 \frac{dt}{dt} = G m_1 d\tau = \frac{G m_1 dt}{\sqrt{1 - G^2 t^2}} \rightarrow p_\tau = G \int d\tau, \quad d\tau = \frac{dt}{\sqrt{1 - G^2 t^2}}, \quad c = 1, \]

As the momentum of the masses grows with time, the mass of the system increases relative to an observer at rest. Evidently,

\[ p_\tau = \frac{m_1 \bar{v}(t)}{\sqrt{1 - \frac{\bar{v}(t)^2}{c^2}}} = \frac{\bar{v}}{\sqrt{1 - \frac{\bar{v}^2}{c^2}}} = G \int d\tau \]

The force in the spatial dimension \( f_s \) can be estimated by Newton's gravitational force law:
\[ f_s = -\sqrt{G} m_1 m_2 \left( \frac{d\tau}{dt} \right)^2 \quad \rightarrow \quad \sqrt{f_s} = i \sqrt{\frac{\sqrt{G} m_1 m_2}{r^2}} \frac{\left( \frac{d\tau}{dt} \right)^2}{r^2 \left( \frac{d\tau}{dt} \right)^2} = i \sqrt{\frac{\sqrt{G} m_1 m_2}{r^2}} = i \sqrt{G}, \]

the factor \((r\tau/dt)^2\) arises because of the spatial expansion due to intrinsic time. Hence,

\[
\left( \sqrt{f_R} \right)^2 = \left( \sqrt{f_s} \right)^2 + \left( \sqrt{f_T} \right)^2 \rightarrow f_R = -\sqrt{G} + G \frac{d\tau}{dt}, \quad m_1, m_2, r = 1,
\]

where \(f_R\) signifies the ‘resultant force’. However, there are two \(f_R\)'s, which point in opposite directions. Therefore, the system has a total force 'f_T' of zero. Moreover,

\[
\langle f_T | d\tau \rangle = 0 \rightarrow E = \text{a constant},
\]

where \(E\) is the ‘total energy’ in the system. The system described above does no work.

If the masses in the problem are also magnets and if a circular copper wire surrounds the system perpendicular to the plane of the page, then the motion of the masses would produce an alternating electric current in the wire.

For those interested in constructing a black hole of this nature, keep in mind it would take about 11.7 million years for the masses to reach a velocity of 1 \([cm/sec]\). And it would take about \(1.2 \times 10^8\) billion years to accelerate them to close to the speed of light.

### 7.8 Concluding Remarks

General relativity arises from image theory through instantiation. However, image theory predicts that Nature possesses an ‘intrinsic time’. Moreover, the existence of intrinsic time removes an invariant ‘proper-time’ and replaces it with 'intrinsic proper-time'.

According to image theory, the distance

\[
ds^2 = \sum_{ij} g_{ij} dx^i dx^j
\]

is a generalization of ‘number interference’. The factor ‘\(g_{ij}\)’ is called the ‘metric tensor’ if the elements of the \(g_{ij}\) have second order continuous derivatives, \(g_{ij}\) is nonsingular, \(g_{ij} = g_{ji}\) and if \(g_{ij}\) is a tensor. Otherwise, \(g_{ij}\) simply indicates the extent of number interference.

Image theory predicts the existence of both dark- and light-proper-times. If intrinsic time is ignored, the difference between the dark- and light-proper-time is a constant. Both light- and dark-proper-times remain invariant along a worldline. The difference between the dark- and light-proper-times is invariant, since their difference is a constant.
Evidently, in a universe devoid of matter and energy, where no ‘intrinsic time’ exists, all objects move along geodesics. However, in a universe containing intrinsic time, objects no longer move along geodesics. There must be a force causing a deviation from a geodesic path. According to image theory, the force arises as a consequence of intrinsic time. If this hypothesis is correct, Einstein’s gravitational equations require modification.

If the general theory of relativity is to be unified with the physics on a small-scale, both $R_{\mu \nu}$ and $T_{\mu \nu}$ must be invertible. However, the tensor '$R_{\mu \nu}$' is not generally invertible. The conditions for invertibility are $T_{\mu \nu} \neq 0$ (no flat spaces), forbidding a universe without matter and energy.
Chapter 8

Image Theory and Quantum Mechanics

“Where misunderstanding dwells, misuse will not be far behind. No theory in the history of science has been more misused and abused by cranks and charlatans—and misunderstood by people struggling in good faith with difficult ideas—than quantum mechanics.”

— Sean Carroll, The Big Picture: On the Origins of Life, Meaning, and the Universe Itself

8.0 Introduction

The theory of relativity emerges from image theory through ‘instantiation’. From these beginnings a new form of light materializes – ‘dark-light’, light that is not directly detectable. Dark-light, however, travels faster than light-light, complicating the process of determining how much time runs off a clock moving with the speed of light-light opposed to one moving with the speed of dark-light. Although the question is purely hypothetical, intuitively, a clock moving with the speed of dark-light should run backwards in time as measured by a stationary clock. The time reversal problem is overcome by realizing that how much time ticks off a moving clock as measured by a stationary clock depends on which light speed, dark- or light-, is used as a standard for specifying velocity.

Image theory predicts a ‘cosmic time’. Cosmic time turns out to be synonymous with spatial expansion. Care must be taken not to confuse spatial expansion with the idea of a ‘space-time distance’, in other words, with ‘proper-time’.

Comparatively, relating image theory to special and general relativity is reasonably straightforward, principally because the foundations of relativity theory are fairly well understood. Abiding by the tenets of relativity theory virtually guarantees that theoretical investigations will rest on more or less solid footing.

By contrast, relating image theory to quantum mechanics treads along precarious ground. It is virtually universally agreed that the theory of relativity has but one interpretation - the ‘space-time’ interpretation. On the other hand, quantum mechanics entertains dozens of interpretations, none of which provide a definitive foundation for the subject. Even its most experienced practitioners and ardent supporters regard quantum mechanics as strangely opaque and, at times, acknowledge that the subject is hostile to commonly accepted tenets. Navigating the theoretical “mine field” that is quantum mechanics exposes a landscape of pitfalls and boobytraps. At times, it is difficult to adequately grasp the implications of a quantum concept, since there is little in the way of a definitive theoretical foundation that supports the subject.
Nevertheless, a reasonably safe bet for a beginning should involve showing how the new theory leads to Schrödinger’s equation or at least some reasonable facsimile, since Schrödinger’s equation is fundamental to quantum mechanics. The first step is a review of how Schrödinger’s equation is derived, followed by a discussion of how it translates into the new theory.

### 8.1 Schrödinger’s Equation Revisited

Quantum mechanics is concerned with the ‘wave function’ of a body, designated ‘Ψ’. Ordinarily, no physical interpretation is given to the wave function, but if Ψ meets a list of prerequisites, then the value ‘ΨΨ†’ is proportional to the probability of finding the body at a specific location at a given moment [45].

#### 8.1.1 Simple Harmonic Motion (Vibration)

Simple harmonic motion is described in terms of oscillations along a stationary axis ‘x’ in time ‘t’, expressed by the two-parameter function

\[
y(t, x) = Ae^{i\omega t}
\]

Since y is a function only of t, x does not change and is arbitrary.

#### 8.1.2 Wave Motion

But remembering that

\[
x = vt \rightarrow t = x/v \rightarrow t - x/v = 0,
\]

if the same condition exists at time ‘t − x/v’ as t = 0, in other words, if y is a periodic function, then

\[
y(t, x) = Ae^{i\omega(t-x/v)},
\]

where y becomes a function of both ‘t’ and ‘x’. If the same state of vibration exists at \( x \neq 0 \) as \( x = 0 \), but with a phase difference corresponding to the velocity ‘v’ of the propagation of the phase, then the motion is called a ‘wave’.

The value of y at ‘x = 0’ repeats if

\[
x = \frac{2\pi v}{\omega} = \frac{v}{\nu},
\]

where \( \omega x/v = 2\pi \) is called the ‘period’ of the wave. The distance, designated ‘λ’, which separates points of equal phase, is called the ‘wave-length’. Letting \( x = \lambda \) in the equation above, then \( \lambda v = v \). Hence, the ‘wave length \times frequency’ is equal to the velocity of the propagation of the phase. In this case, the direction of motion is the +x-direction.
8.1.3 The Wave Equation

If \( y(t, x) \) is differentiated twice with respect to \( x \) and twice with respect to \( t \), then

\[
\frac{\partial^2 y}{dt^2} = -\omega^2 A e^{i\omega(t-x/v)}, \quad \frac{\partial^2 y}{dx^2} = -\frac{\omega^2}{v^2} A e^{i\omega(t-x/v)} \rightarrow v^2 \frac{\partial^2 y}{dx^2} = -\omega^2 A e^{i\omega(t-x/v)}
\]

Equating the two equations gives

\[
\frac{\partial^2 y}{dx^2} = \frac{1}{v^2} \frac{\partial^2 y}{dt^2},
\]

which is called the ‘wave equation’. It describes a freely moving wave, one not influenced by outside forces.

8.1.3.1 Solution to the Wave Equation

The most general solution ‘\( y \)’ to the wave equation can be written

\[
y = A \cos \omega \left( t - \frac{x}{v} \right) - iA \sin \omega \left( t - \frac{x}{v} \right)
\]

Note that \( y \) is a complex function of \( x \) and \( t \). However, the only physically meaningful part of \( y \) is either the imaginary or the real part, one or the other, but not both. Customarily, the imaginary part of \( y \) is discarded as physically irrelevant.

In quantum mechanics, \( y \) is typically replaced by ‘\( \Psi \)’ symbolizing the fact that \( \Psi \), by itself, is physically meaningless, but complex. Under the Copenhagen interpretation of quantum mechanics, complex functions have no physical significance.

Since \( \omega = 2\pi \nu \) and \( v = \lambda \nu \),

\[
\Psi(t, x) = A e^{i2\pi \nu(t-x/v)} = A e^{i2\pi(\nu t-x\nu/v)} = A e^{i2\pi(\nu t-x/\lambda)}
\]

8.1.4 The Derivation of Schrödinger’s Equation

Remembering from fundamental quantum mechanics that \( E = \hbar \nu \) and \( \lambda = \hbar / p \), where \( p \) is the momentum, then

\[
\Psi(t, x) = A e^{i\frac{2\pi}{\hbar}(Et-px)}
\]

The function ‘\( \Psi \)’ is the mathematical description of an unrestricted particle having energy ‘\( E \)’ and momentum ‘\( p \)’ moving in the +x-direction. Of course, this is not strictly true, since \( \Psi \) does not describe an actual physical wave, but a wave of probability. However, ‘\( \Psi \Psi^\dagger \)’ is proportional to the probability of finding the particle at position ‘\( x \)’ at time ‘\( t \)’.
If \( \Psi \) is differentiated twice with respect to \( x \) and once with respect to \( t \), then
\[
\frac{\partial^2 \Psi}{\partial x^2} = -\frac{4\pi^2 p^2}{h^2} \Psi \rightarrow p^2 \Psi = -\frac{h^2}{4\pi^2} \frac{\partial^2 \Psi}{\partial x^2}, \quad \frac{\partial \Psi}{\partial t} = \frac{2\pi i E}{h} \Psi \rightarrow E \Psi = \frac{h}{2\pi i} \frac{\partial \Psi}{\partial t}
\]

At speeds small compared to the speed of light, Hamilton's principle can be applied:
\[
E = K.E. + P.E.,
\]
where \( K.E. \) signifies the 'kinetic energy' and \( P.E. \) the 'potential energy' of the wave. Hence,
\[
E = \frac{p^2}{2m} + V(x, t) \rightarrow E \Psi = \frac{p^2 \Psi}{2m} + V(x, t) \Psi, \quad K.E. = \frac{p^2}{2m}, \quad P.E. = V(x, t)
\]

Substituting the values for \( E \Psi \) and \( p^2 \Psi \) into the equation above gives
\[
\frac{h}{2\pi i} \frac{\partial \Psi}{\partial t} = -\frac{h^2}{8\pi^2 m} \frac{\partial^2 \Psi}{\partial x^2} + V \Psi \rightarrow \frac{i h}{2\pi} \frac{\partial \Psi}{\partial t} = \frac{h^2}{8\pi^2 m} \frac{\partial^2 \Psi}{\partial x^2} - V \Psi,
\]
which is the time-dependent form of Schrödinger's equation [45].

### 8.2 A Facsimile of Schrödinger's Equation within Image Theory

An attempt at deriving a reasonable facsimile to Schrödinger's equation encounters immediate difficulties. In image theory, what is hypothesized as Planck's constant \( \sqrt{\hbar} \), is not a constant, but an instantiated limit. Moreover,
\[
\sqrt{\hbar} \approx 8.45 \times 10^{-27} \left[ \sqrt{1/sec} \right],
\]
which has different dimensional units than the quantum mechanical \( \hbar \), suggesting that \( \hbar \) is what physically vibrates. Moreover,
\[
\sqrt{\hbar_i} \approx h_q,
\]
where \( h_q \) denotes Planck's constant. Within quantum mechanics, the quantity \( h_q \) is generally considered a constant of proportionality. For example, the quantum mechanical equation for the energy in a wave is given by
\[
E = h_q [gm - cm^2/sec] v [1/sec]
\]
The ontology of quantum mechanics maintains that "amounts of energy" in a system come in integer multiples of \( h_q \). Quantum mechanics defines energy '\( E \) in terms of an 'energy operator'
\[
\hat{H} = i h_q \frac{\partial}{\partial t},
\]
which acts on $\Psi(x, t)$. The wave function “$\Psi$” represents all that is known about a quantum system. All observables and measurements in quantum mechanics are associated with ‘operators’. There is a different operator for each observable, one for location, one for momentum, one for energy and so on. Measurements are represented by eigenvalue equations. For instance,

$$\hat{H}\Psi = E\Psi$$

is the eigenvalue equation for computing the energy of the system. When in a certain state, if the wave function is operated on by $\hat{H}$, then, at location ‘$x$’ and time ‘$t$’, the system will definitely have energy ‘$E$’. Although $\Psi$ is a complex function of $x, t$, all operators in quantum mechanics are Hermitian, meaning that all the values representing measurements and observations are real numbers.

By contrast, image theory suggests that $h_i$ is the entity that actually vibrates, raising the immediate problem of finding a common collection of properties and relations between image theory and quantum mechanics. However, through a very fortunate set of circumstances, the problem is tractable. To see this, let

$$h_i(t) = \omega(t) \rightarrow \sqrt{h_i(t)} = \sqrt{\omega(t)}$$

and let

$$\sqrt{\theta} = \sqrt{\omega} \left( \sqrt{t} - \frac{x}{\sqrt{v}} \right) = \sqrt{h_i} \left( \sqrt{t} - \frac{x}{\sqrt{v}} \right), \quad t, \frac{x}{v} \geq 0,$$

and consider the instantiation of $e^{\sqrt{\theta}}$. If the instantiation of $e^{\sqrt{\theta}}$ is independent of $t$, then $\theta$ can be an arbitrary function of $t$. However, if the instantiation of $e^{\sqrt{\theta}}$ depends on $t$, then since

$$\sqrt{\theta} = \sqrt{h_i} \left( \sqrt{t} - \frac{x}{\sqrt{v}} \right),$$

and $h_i$ is a function of $t$, the instantiation of $e^{\sqrt{\theta}}$ would proceed as a unit, meaning that $h_i$ and $\sqrt{t} - \sqrt{x/v}$ could not be instantiated independently.

Recall that if

$$f(\theta) = \left( \frac{1}{\sqrt{e}} \right)^{1-\sqrt{\theta}} (\sqrt{e})^{\sqrt{\theta}},$$
expanding $f$ in a power series, gives

$$f(\theta) = e^{-\frac{1}{2}} + e^{-\frac{1}{2}\theta} + \frac{1}{2} e^{-\frac{1}{2}\theta} + \frac{1}{6} e^{-\frac{1}{2}\theta^{3/2}} + \frac{1}{24} e^{-\frac{1}{2}\theta^2} + \frac{1}{120} e^{-\frac{1}{2}\theta^{5/2}} + \frac{1}{720} e^{-\frac{1}{2}\theta^3} + \frac{1}{5040} e^{-\frac{1}{2}\theta^{7/2}} + \ldots$$

$$= e^{-\frac{1}{2}} \left(1 + \sqrt{\theta} + \frac{1}{2} \theta + \frac{1}{6} \theta^{3/2} + \frac{1}{24} \theta^2 + \frac{1}{120} \theta^{5/2} + \frac{1}{720} \theta^3 + \frac{1}{5040} \theta^{7/2} + \ldots\right) \to \sqrt{e} f(\theta) = \sqrt{e} \left(\frac{1}{\sqrt{e}}\right)^{1-\sqrt{\theta}} (\sqrt{e})^{\sqrt{\theta}}$$

$$= 1 + \sqrt{\theta} + \frac{1}{2} \theta + \frac{1}{3} \theta^{3/2} + \frac{1}{4} \theta^2 + \frac{1}{5} \theta^{5/2} + \frac{1}{6} \theta^3 + \frac{1}{7} \theta^{7/2} + \ldots = e^{\sqrt{\theta}}$$

Note that

$$\cos(i\sqrt{\theta}) = 1 + \frac{1}{2!} \theta + \frac{1}{4!} \theta^2 + \frac{1}{6!} \theta^3 + \ldots,$$

$$-i \sin(i\sqrt{\theta}) = \sqrt{\theta} + \frac{1}{3!} \theta^{3/2} + \frac{1}{5!} \theta^{5/2} + \frac{1}{7!} \theta^{7/2} + \ldots$$

Hence,

$$\sqrt{e} f(\theta) = \sqrt{e} \left(\frac{1}{\sqrt{e}}\right)^{1-\sqrt{\theta}} (\sqrt{e})^{\sqrt{\theta}} = e^{\sqrt{\theta}} = \cos(i\sqrt{\theta}) - i \sin(i\sqrt{\theta})$$

Let

$$a = \cos(i\sqrt{\theta}) = \cosh \sqrt{\theta} \to \cosh^{-1} a = \sqrt{\theta} \to \cosh^{-2} a = \theta \to \cosh^2(\cosh^{-2} a) = a = \cosh^2(\theta) \to \sqrt{a} = \pm \cosh \theta$$

Similarly, let

$$b = -i \sin(i\sqrt{\theta}) = \sinh \sqrt{\theta} \to \sinh^{-1} b = \sqrt{\theta} \to \sinh^{-2} b = \theta \to \sinh^2(\sinh^{-2} b) = b = \sinh^2(\theta) \to \sqrt{b} = \pm \sinh \theta, \quad \theta \geq 0$$

Therefore,

$$\sqrt{b} = \sinh \theta = -i \sin(i\theta), \quad \sqrt{a} = \cosh \theta = \cos(i\theta), \quad \theta \geq 0$$

Hence, in instantiation space,

$$\sqrt{\sqrt{e} f(\theta)} = (\sqrt{a}, \sqrt{b}) \sqrt{a + b} = (\cos(i\theta), -\sin(i\theta)) \sqrt{\cos^2(i\theta) + \sin^2(i\theta)} = \cos(i\theta) - i \sin(i\theta),$$

since $\cos^2(i\theta) + \sin^2(i\theta) = 1$. Let $i\theta = y$, then

$$\cos(i\theta) - i \sin(i\theta) = \cos y - i \sin y = e^{-iy} = e^{-i(i\theta)} = e^\theta, \quad \theta \geq 0$$
This implies that

\[ e^{\sqrt{\theta}} \rightarrow e^\theta \]

In other words, \( e^{\sqrt{\theta}} \) is instantiated by \( e^\theta \). To be more in line with quantum mechanics, \( e^\theta \) should be complex. Recall that

\[ \sqrt{e f(\theta)} = \cos(i\theta) - i \sin(i\theta) \]

Replacing \( \theta \) with \( -i\theta \),

\[ \sqrt{e f(\theta)} = \cos i(-i\theta) - i \sin i(-i\theta) = \cos \theta - i \sin \theta = e^{-i\theta} \]

Hence,

\[ e^{\sqrt{\theta}} \rightarrow e^{-i\theta} \]

In other words, \( e^{\sqrt{\theta}} \) is instantiated by \( e^{-i\theta} \). To see this in more detail,

\[
\begin{align*}
\{ & e^{\sqrt{\theta}} \rightarrow e^{-i\theta}, \quad \text{if } \theta \geq 0 \\
& e^{-\sqrt{\theta}} \rightarrow e^{i\theta}, \quad \text{if } \theta < 0 
\end{align*}
\]

In the case shown above, a non-periodic function is mapped to a periodic function. Evidently, if the instantiations are to be valid, \( \theta \) must be restricted to intervals of \( 2\pi \).

Moreover, the instantiation of \( \sqrt{e f(\theta)} \) is independent of \( t \). Therefore, \( \theta \) can be an arbitrary function of \( t \). Let

\[ \theta = h_i \left( \sqrt{t} - \sqrt{\frac{x}{v}} \right)^2 = h_i \left( t + \frac{x}{v} - 2 \sqrt{\frac{x t}{v}} \right), \quad \sqrt{\frac{x t}{v}} = \begin{cases} \sqrt{\frac{x t}{v}}, & \frac{x t}{v} \geq 0 \\
-\sqrt{\frac{x t}{v}}, & \frac{x t}{v} < 0 \end{cases} \]

Ordinarily, the next step in the process would be to separate the variables ‘\( x \)’ and ‘\( t \)’ thereby isolating the energy ‘\( E \)’, associated with time ‘\( t \)’, from the momentum ‘\( p \)’, associated with space ‘\( x \)’. But, in this case, pursuing such an approach leads to immediate defeat. The interference term ‘\( 2\sqrt{x t / v} \)’ precludes separating \( x \) from \( t \). There is no way of determining the energy independently from the momentum of the system. In other words, there is no way of separating space from time. It is only meaningful within image theory to discuss ‘space-time’. 
8.3 Instantiations and Quantum Mechanics

The position operator ‘\( \hat{X} \)’ in quantum mechanics is defined as

\[
\hat{X} |\Psi\rangle = x |\Psi\rangle,
\]

where ‘\( \hat{X} \)’ simply multiplies the function '\( \Psi \)' by its location on a line. If the ‘eigenvalues’ and ‘eigenvectors’ of \( \hat{X} \) are ‘\( \lambda \)’ and ‘\( |\Psi\rangle \)’ respectfully, then

\[
\hat{X} |\Psi\rangle = \lambda |\Psi\rangle \rightarrow x |\Psi\rangle = \lambda |\Psi\rangle \rightarrow (x - \lambda) |\Psi\rangle = |0\rangle
\]

8.3.1 The Dirac \( \delta \)-Function

The Dirac delta function is the solution to the equation above. In other words,

\[
\Psi(x) = \delta(x - x_0),
\]

where

\[
\delta(x - x_0) = \begin{cases} 
1, & \text{if } x = x_0 \\
0, & \text{if } x \neq x_0
\end{cases}
\]

\[
= \frac{x_0 + \frac{\varepsilon}{2}}{\varepsilon} - \frac{x_0 - \frac{\varepsilon}{2}}{\varepsilon} = \frac{\varepsilon}{\varepsilon} = 1, \quad \delta(x) = \frac{1}{\varepsilon}
\]

The vector ‘\( |\Psi\rangle = |\delta(x - \lambda)\rangle \)’ is an eigenvector of \( \hat{X} \) with eigenvalue ‘\( \lambda \)’. There are an infinite number of eigenvectors, since there are an infinite number of positions along the real line. The eigenvectors ‘\( |\delta(x - \lambda)\rangle \)’ are orthogonal (not normalized), since

\[
\langle \delta(x - \lambda) | \delta(x - \lambda') \rangle = \int \delta^\dagger_{\lambda'} \delta_{\lambda'} dx = 0, \quad \lambda \neq \lambda'
\]

Now

\[
\langle \delta(x - \lambda) | \Psi \rangle = \int \delta(x - \lambda) \Psi(x) dx = \int \delta(x - \lambda) \Psi(\lambda) dx = \Psi(\lambda) \int \delta(x - \lambda) dx = \Psi(\lambda),
\]

since

\[
\int \delta(x - \lambda) dx = 0
\]
except where \( x = \lambda \), then
\[
\int \delta(x) \, dx = 1
\]
If \( x = \lambda \),
\[
\langle \delta(x - \lambda) | \Psi \rangle = \Psi(x)
\]
Moreover, the probability \( P_x \) that the particle is found at position \( x \) is
\[
P_x = \langle \delta(x - \lambda) | \Psi \rangle \langle \Psi | \delta(x - \lambda) \rangle = \psi^\dagger \psi \rightarrow \int \psi \psi^\dagger \, dx = 1
\]
For instance, if a particle is confined to a line \( 1 \, [cm] \) in length, then
\[
\int_0^1 \psi^\dagger \psi \, dx = 1,
\]
which indicates that it is certain that the particle is somewhere on the line. If the particle moves unconstrained along the line, then, for example,
\[
\int_0^{1/2} \psi^\dagger \psi \, dx = \frac{1}{2}
\]
In other words, the probability that the particle will be found in the interval \( 0 \leq x \leq 1/2 \) is \( 1/2 \). Notice that
\[
\int_x^\infty \psi^\dagger \psi \, dx = 0
\]
The probability that the particle is at any particular position along the line is zero, which implies that the particle is somewhere on the line, but nowhere in particular. This is a very strange interpretation, since if an attempt is made to find the particle at position \( x \), quantum physics predicts it will never be found there.

### 8.3.2 Position Instantiation

Image theory interprets this situation differently. If a particle is at position \( x \), then this is instantiated by creating an image for \( x \) and then forming a function \( f \) such that \( x \xrightarrow{f} x \). In this case, \( x \) is a real number and there is no interference if the position of a particle is projected into instantiation space. In other words, a freely moving particle is one that cannot interfere with itself. If \( x \geq 0 \), then
\[
\langle x, 0 \rangle \xrightarrow{f} (\sqrt{x}, 0) \sqrt{x + 0} = x \xrightarrow{i} ix
\]
If $x > 0$, then $-x < 0$ and $(-x, 0) \overset{f}{\rightarrow} -x$. To see this,

$$(-x, 0) \overset{f}{\rightarrow} (\sqrt{-x}, 0)\sqrt{-x} + 0 = i^2 \sqrt{-x} = -x \rightarrow -x \overset{f}{\rightarrow} -ix$$

In general,

$$x \overset{f}{\rightarrow} ix$$

### 8.3.2.1 The Physical Interpretation of Position Instantiation

In image theory, it is always possible to know the position of a particle. So,

$$\int_{x}^{x} dx = 0 \rightarrow i \int_{x}^{x} dx = i0$$

is interpreted as the particle is definitely at 'x'. And

$$\int_{0}^{x} dx = x \rightarrow i \int_{0}^{x} dx = ix$$

might be interpreted as the particle has moved from position '0' to position 'x', a distance 'x'.

If

$$\int_{0}^{x} dx = x < 0 \rightarrow i \int_{0}^{x} dx = ix < i0,$$

the particle moved a distance in the negative direction.

### 8.3.3 Momentum Instantiation

Momentum in quantum mechanics is proportional to $\partial \Psi / \partial x$. Consider the eigenvalue equation

$$\hat{R} \Psi(x) = k \Psi(x),$$

where $\hat{R} = -i \partial / \partial x$ and $k$ is the 'eigenvalue' of $\hat{R}$. The solution $'\Psi(x)'$ to this equation is

$$\Psi(x) \propto e^{ikx}$$

Note that

$$\partial \Psi / \partial x = ik \Psi \rightarrow -i \partial \Psi / \partial x = k \Psi$$
Moreover,

\[ \Psi(x) = ae^{ikx} = \alpha \cos(kx) + i\alpha \sin(kx), \quad \alpha \in \mathbb{C} \]

is a more general solution. It follows that

\[ \Psi^\dagger \Psi = \alpha^\dagger e^{-ikx} \alpha e^{ikx} = \alpha^\dagger \alpha e^{-ikx} x e^{ikx} = \alpha^\dagger \alpha \]

Since the period of \( \sin(kx), \cos(kx) \) is \( 2\pi \), \( x \) must change by \( \lambda \) before the wave repeats a cycle. In other words, if \( \lambda \) is the 'wavelength', then

\[ k\lambda = 2\pi \rightarrow \lambda = \frac{2\pi}{k} \]

Recall de Broglie's equation \( p = h_q/\lambda \), where \( p \) is the momentum of a particle. Note that

\[ p = \frac{h_q}{\lambda} = h_q k, \quad h_q = \frac{h_q}{2\pi} \]

The momentum of a particle is proportional to the eigenvalue \( k \). Moreover, since

\[ \Psi^\dagger \Psi = \alpha^\dagger \alpha, \]

then \( \alpha^\dagger \alpha \) is proportional to the probability of finding the particle at position \( x \), if its momentum is \( p = h_q k \).

If \( \alpha = 1 \), then \( \Psi^\dagger \Psi = 1 \), which indicates that the particle is somewhere on the line. Quantum mechanics requires that if the momentum of the particle is known, the position of the particle spreads out over the whole line. The only information available about the position of the particle is that it is somewhere on the line, but nowhere in particular.

Image theory treats this situation differently. Let

\[ \Psi \propto e^{i\theta} \]

Image theory requires that \( \Psi \) be instantiated. Therefore,

\[ e^{i\theta} \rightarrow (\cos(i\theta), -\sin(i\theta))\sqrt{\sin^2(i\theta) + \cos^2(i\theta)} = (\cos(i\theta), -\sin(i\theta)) = e^{i\theta} \rightarrow e^{i\theta} \rightarrow e^\theta \]

\[ \rightarrow e^\theta \rightarrow e^{-i\theta} = \bar{\Psi}, \quad \theta \rightarrow -i\theta, \quad \theta = -kx \]

In other words,

\[ \Psi \rightarrow \bar{\Psi} \]
8.3.3.1 The Physical Interpretation of the Instantiation of Momentum

The ontology of quantum mechanics assigns \( x, t \) values in terms of the energy and momentum of a particle, accomplished through the relationships

\[ E = h\nu \rightarrow Et = hvt, \quad x = \lambda = \frac{h}{p} \]

However, image theory does not allow \( x \) and \( t \) to be separated into the distinct concepts of 'space' and 'time'. As such, a different set of relationships must be established between the energy and momentum of a particle than those found in quantum mechanics. Moreover, those relationships must be instantiated. As a matter of hypothesis, define

\[ \sqrt{-i\hbar i\lambda} = \sqrt{E}, \]

where \( \sqrt{E} \) is the 'instantiated' energy. It follows that

\[ \sqrt{\hbar i\lambda} = \sqrt{E} \rightarrow \hbar i\lambda = E \rightarrow h_i\lambda = \nu = \frac{E}{p} \]

8.4 Deriving a Substitute for Schrödinger’s Equation

Let

\[ \Psi = \sqrt{A}e^{\sqrt{\theta}} \rightarrow \overline{\Psi} = Ae^{-i\theta} = Ae^{-ih_i(\sqrt{t} - \sqrt{x/v})^2}, \quad \theta = h_i\left(\sqrt{t} - \sqrt{x/v}\right)^2 \]

Differentiating \( \overline{\Psi} \) once with respect to \( x \) and once with respect to \( t \) leaves

\[ \frac{\partial \Psi}{\partial x} = ih_i \frac{\left(\sqrt{t} - \sqrt{x/v}\right)}{\sqrt{x/v}} \overline{\Psi}, \quad \frac{\partial \overline{\Psi}}{\partial t} = -ih_i \frac{\left(\sqrt{t} - \sqrt{x/v}\right)}{\sqrt{t}} \overline{\Psi} \]

Hence,

\[ \sqrt{x/v} \frac{\partial \Psi}{\partial x} = ih_i \left(\sqrt{t} - \sqrt{x/v}\right) \overline{\Psi}, \quad -\sqrt{t} \frac{\partial \overline{\Psi}}{\partial t} = ih_i \left(\sqrt{t} - \sqrt{x/v}\right) \overline{\Psi} \rightarrow \frac{\sqrt{x/v} \left(\partial \Psi / \partial x\right)}{\sqrt{t}} = -\sqrt{t} \frac{\partial \overline{\Psi}}{\partial t} \rightarrow \left(\partial \Psi / \partial x\right) = -\frac{1}{v} \frac{\partial \overline{\Psi}}{\partial t}, \]

\[ \sqrt{x/v} = \sqrt{t} \]

Remembering that \( \nu = E/p \), then

\[ \frac{E}{p} \left(\frac{\partial \Psi}{\partial x}\right) = -\frac{\partial \overline{\Psi}}{\partial t} \rightarrow \frac{1}{p} \left(\frac{\partial \Psi}{\partial x}\right) = -\frac{1}{E} \frac{\partial \Psi}{\partial t} \]
8.4.1 The Interpretation of $\Psi$

Recall that

$$\frac{\partial \bar{\Psi}}{\partial x} = ith \frac{\sqrt{t} - \sqrt{x}}{\sqrt{v} \sqrt{v}} \bar{\Psi} \rightarrow v \frac{\partial \bar{\Psi}}{\partial x} = ith \frac{\sqrt{t} - \sqrt{x}}{\sqrt{v}} \bar{\Psi} \rightarrow -\frac{\partial \bar{\Psi}}{\partial t} = ith \frac{\sqrt{t} - \sqrt{x}}{\sqrt{v}} \bar{\Psi}$$

Since $v = E/p$, then

$$-\frac{\partial \bar{\Psi}}{\partial t} = ith \frac{\sqrt{t} - \sqrt{xp}}{\sqrt{xp} \sqrt{E}} \bar{\Psi} \rightarrow -\sqrt{xp} \frac{\partial \bar{\Psi}}{\partial t} = ith(\sqrt{Et} - \sqrt{xp}) \bar{\Psi} \rightarrow -\sqrt{xp} \frac{\partial \bar{\Psi}}{ith} \bar{\Psi} = \sqrt{Et} \bar{\Psi} - \sqrt{xp} \bar{\Psi}$$

$$\rightarrow -\sqrt{xp} \frac{\partial \bar{\Psi}}{ith} + \sqrt{xp} \bar{\Psi} = \sqrt{Et} \bar{\Psi} \rightarrow -\frac{\sqrt{xp}}{ith} \left( \frac{1}{ith} \frac{\partial \bar{\Psi}}{\partial t} - \bar{\Psi} \right) = \sqrt{E} \bar{\Psi}$$

Letting $\sqrt{xp}/t = \sqrt{E}$, then

$$-\sqrt{E} \left( \frac{\partial \bar{\Psi}}{ith} - \bar{\Psi} \right) = \sqrt{E} \bar{\Psi} \rightarrow -\frac{1}{ith} \frac{\partial \bar{\Psi}}{\partial t} + \bar{\Psi} = \bar{\Psi} \rightarrow \frac{\partial \bar{\Psi}}{\partial t} = 0 \rightarrow \bar{\Psi} = K, \quad K = \text{a constant}$$

And

$$\sqrt{xp}/t = \sqrt{ph\lambda} \rightarrow x = h\lambda t$$

Moreover, if $\bar{\Psi} = K$, then

$$-\frac{\partial \bar{\Psi}}{\partial t} = ith \frac{\sqrt{t} - \sqrt{xp}}{\sqrt{xp} \sqrt{E}} \bar{\Psi} \rightarrow 0 = \sqrt{t} - \sqrt{xp} \rightarrow Et = xp \rightarrow t = \frac{px}{E}$$

Remembering the relationships

$$\sqrt{ph\lambda} = \sqrt{E} \rightarrow ph\lambda = E \rightarrow p\nu = E \rightarrow \nu = \frac{E}{p}$$

Hence,

$$\theta = h\frac{t + \frac{\chi}{v} - 2\frac{\sqrt{xt}}{\sqrt{v}}}{h} = h\left( \frac{px}{E} + \frac{px}{E} - 2\frac{p^{2}x^{2}}{E^{2}} \right) = 0 \rightarrow e^{-i\theta} = 1$$
8.4.2 Determining Position and Momentum Simultaneously

If position and momentum are determined by projection into instantiation space simultaneously, then

\[
x \rightarrow ix, \quad \Psi \rightarrow \bar{\Psi} = e^{-i\theta}, \quad ix\bar{\Psi} = ixe^{-i\theta} = ix \rightarrow x
\]

If the position of a particle is known to be 'x', the momentum of the particle is arbitrary.
This also true of momentum. If the momentum is known to be 'p', assuming that p can be instantiated by \(-ip\), then

\[
x \rightarrow ix, \quad p \rightarrow -ip, \quad -ixp = x \frac{Et}{x} = Et
\]

If the momentum is known to be p, the position of the particle is arbitrary. Momentum and position cannot be determined simultaneously. In this sense, image theory aligns with quantum mechanics.

8.4.3 Energy Levels in Image Theory

If the particle is moving along a circular path, and if the circle is cut at a point and laid flat, then, at one end, \(x = 0\), and, at the other end, \(x = 2\pi r\), the circumference of the circle. The functions \(\bar{\Psi}(x)\) that define this space have the property

\[
\bar{\Psi}(x) = \bar{\Psi}(x \pm 2\pi r)
\]

Let

\[
\bar{\Psi}(r) = e^{ik(r+2\pi r)}
\]

Recall that

\[
ph_i\lambda = E = ph_i \frac{2\pi}{k} \rightarrow k = \frac{2\pi ph_i}{E}, \quad \lambda = \frac{2\pi}{k}
\]

Hence,

\[
\bar{\Psi}(r) = e^{ik(r+2\pi r)} = e^{l\frac{2\pi ph_i}{E}(r+2\pi r)}
\]

Since \(\bar{\Psi}\) is periodic,

\[
\bar{\Psi}(r) = \bar{\Psi}(r + 2\pi r) \rightarrow e^{l\frac{2\pi ph_i}{E}(r+2\pi r)} = e^{l\frac{2\pi ph_i}{E}r} e^{l\frac{2\pi ph_i}{E}(2\pi r)} = e^{l\frac{2\pi ph_i}{E}r} \rightarrow e^{l\frac{2\pi ph_i}{E}(2\pi r)} = 1
\]
The factor \( e^{i(2\pi ph_i/E)2\pi r} \) is the necessary and sufficient condition ensuring that \( \bar{\Psi} \) is periodic. For any exponential, \( e^{i2\pi n} = 1, \ n \in \mathbb{Z} \). Therefore, in order for \( \bar{\Psi} \) to be periodic,

\[
\frac{4\pi^2 ph_ir}{E} = 2\pi n \rightarrow \frac{2\pi ph_ir}{E} = n \in \mathbb{Z} \rightarrow nE = 2\pi ph_ir, \quad n \in \mathbb{Z},
\]

which shows that the energy of the moving particle is an integer multiple of \( 2\pi ph_ir \). Like quantum mechanics, the energy levels of systems in image theory are quantized.

### 8.5 Image Theory and the Quantum Field

Mathematically, a field in QFT is a collection of harmonic oscillators. Specifically, the energy in a field is the energy in a collection of harmonic oscillators. The total energy ‘\( E \)’ in a harmonic oscillator is

\[
E = K.E. + P.E. \rightarrow E = \frac{p^2}{2m} + \frac{m}{2} \omega^2 x^2, \quad \omega = \sqrt{\frac{k}{m}} \rightarrow \omega^2 = \frac{k}{m}
\]

where \( k \) is the ‘spring constant’. To associate the equation above with the energy of a wave, write

\[
E = \frac{p^2}{2m} + \frac{1}{2} kx^2 \rightarrow E = \frac{p^2}{2m} + \frac{1}{2} \omega^2 x^2, \quad m = 1, \quad \omega^2 = k
\]

In image theory, \( \omega = h_i \), hence,

\[
E = \frac{1}{2} (p + i h_i x)(p - i h_i x) \rightarrow \sqrt{E} = \sqrt{\frac{2}{2}} (p + i h_i x), \quad \sqrt{E^+} = \sqrt{\frac{2}{2}} (p - i h_i x)
\]

Let

\[
\hat{a}^+ = \frac{p + i h_i x}{\sqrt{2h_i}} = \left( \frac{p}{\sqrt{2h_i}}, \frac{h_i x}{\sqrt{2h_i}} \right), \quad \hat{a}^- = \frac{p - i h_i x}{\sqrt{2h_i}} = \left( \frac{p}{\sqrt{2h_i}}, -\frac{h_i x}{\sqrt{2h_i}} \right)
\]

If \( \hat{a}^+ \) and \( \hat{a}^- \) are projected into instantiation space,

\[
|\bar{\hat{a}}^+| = \left| \frac{p}{\sqrt{2h_i}}, \frac{h_i x}{2h_i} \right| = \frac{1}{\sqrt{2h_i}} \sqrt{p^2 + (h_i x)^2} + 2|ph_i x|,
\]

\[
|\bar{\hat{a}}^-| = \left| \frac{p}{\sqrt{2h_i}}, -\frac{h_i x}{2h_i} \right| = \frac{1}{\sqrt{2h_i}} \sqrt{p^2 + (h_i x)^2} - 2|ph_i x|
\]

In this case, both \( |\bar{\hat{a}}^+| \) and \( |\bar{\hat{a}}^-| \) show interference terms in momentum and position. To simplify matters, let \((h_i x)^2 \ll 1\), then
\[ |\tilde{a}^+| |\tilde{a}^-| \approx \left( \frac{1}{\sqrt{2\hbar}} \sqrt{p^2 + 2p\hbar x} \right) \left( \frac{1}{\sqrt{2\hbar}} \sqrt{p^2 - 2p\hbar x} \right) \to (|\tilde{a}^+| |\tilde{a}^-|)^2 \]

\[ = \left( \frac{1}{2\hbar} \right)^2 (p^2 + 2p\hbar x)(p^2 - 2p\hbar x) \to |\tilde{a}^+| |\tilde{a}^-| \approx \frac{1}{2\hbar} \sqrt{p^4 - (2p\hbar x)^2}, \]

\[(h\hbar x)^2 \ll 1, \]

where the term \((h\hbar x)^2\) has been discarded as too small to count.

Recall that the energy in a harmonic oscillator is

\[ \hat{H} = \hbar \omega \hat{a}^+ \hat{a}^- = \hbar \hbar \omega \hat{a}^+ \hat{a}^- \to |H| = \hbar \hbar |\tilde{a}^+| |\tilde{a}^-| \approx \frac{\hbar}{2} \sqrt{p^4 - (2p\hbar x)^2} \]

\[ = \hbar \sqrt{\frac{p^4}{4} - (p\hbar x)^2}, \quad \omega = \hbar \]

As a result of the interference term \((p\hbar x)^2\), the kinetic and potential energies cannot be separated, and hence, cannot be known simultaneously.

### 8.5.1 The Quantum Field in Image Theory

If there are a number of harmonic oscillators, then

\[ \hat{a}_j^+ = \frac{p_j + i\hbar x_j}{\sqrt{2\hbar}}, \quad \hat{a}_j^- = \frac{p_j - i\hbar x_j}{\sqrt{2\hbar}} \]

The wave function \(\psi(x)\) in quantum mechanics takes the form \(\psi(x) \propto e^{ikx}\), where \(k\) can be positive or negative. If the wave is travelling in a closed loop, with the distance around the loop signified by \(S\), then \(e^{iks} = 1\), since

\[ e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1, \]

the angle around any closed loop being \(2\pi\). Therefore,

\[ kS = 2\pi m, \quad m \in \mathbb{N} \to k = \frac{2\pi m}{S} \]

Note that \(S/m = \lambda\) is the wave length, so that

\[ k\lambda = 2\pi \to k = \frac{2\pi}{\lambda} \]

\[ f(x) = f(x + p) \]

\[ a \quad \text{\(\sim\)} \quad b \quad \text{\(\sim\)} \quad \text{\(\sim\)} \]

Figure 8.4.1-1
If there is a function \( f(x) \) continuous in an interval \([a, b]\) and \( f(a) = f(b) \), then \( f \) is 
'periodic' with period \('p'\), if the cycle repeats itself (see Figure 8.4.1-1). What Fourier understood was that \( f \) could always be written in terms of the sine or cosine function, in other words, in terms of waves-like functions.

A field \( \Psi(x) \), in classical physics, is defined

\[
\Psi(x) = \sum_j \alpha_j e^{ijx}, \quad \Psi^\dagger(x) = \sum_j \alpha_j^\dagger e^{-ijx},
\]

where \( \Psi(x) \) and \( \Psi^\dagger(x) \) are Fourier transforms of each other (see Book II: Sec. 16.8.1).

To see this in more detail, recall that \( \psi(x) \propto e^{ikx} \), where \( k = \frac{2\pi m}{S} \), \( m \in N \) and \( S \) is the distance around a closed loop. Note that

\[
\int_{-S/2}^{S/2} e^{ikx} \, dx = \begin{cases} S \quad & \text{if } k = 0 \\ 0 \quad & \text{if } k \neq 0 \end{cases} \rightarrow \int_{-S/2}^{S/2} e^{ikx} \, dx = 2\pi \delta(k)
\]

To see this, recall that the Dirac delta function with width of \( '2\pi/S' \) has height \( 'S' \).

Hence, \( 2\pi S/S = 2\pi \) and if \( k \neq 0 \), then

\[
\int_{-S/2}^{S/2} e^{ikx} \, dx = \frac{1}{ik} e^{ikx} \bigg|_{-S/2}^{S/2} = \frac{1}{ik} [\cos kx + i \sin kx] \bigg|_{-S/2}^{S/2} = \frac{1}{ik} \left( \left[ \cos \left( \frac{kS}{2} \right) + i \sin \left( \frac{kS}{2} \right) \right] - \left[ \cos \left( -\frac{kS}{2} \right) + i \sin \left( -\frac{kS}{2} \right) \right] \right) = \frac{2}{k} \sin \left( \frac{kS}{2} \right) = 0,
\]

\( m \in N \), \( kS = 2\pi m \)

Therefore,

\[
\lim_{S \to \infty} \int_{-S/2}^{S/2} e^{ikx} \, dx = \int_{-\infty}^{\infty} e^{ikx} \, dx = 2\pi \delta(k) \rightarrow \int_{-\infty}^{\infty} e^{ikx} \, dk = 2\pi \delta(x)
\]

Suppose, in the equations for \( \Psi \) and \( \Psi^\dagger \), the \( \alpha_j \)'s are replaced by \( \hat{\alpha}^- \) and \( \hat{\alpha}^+ \), then

\[
\hat{\Psi}(x) = \sum_k \hat{\alpha}_k^- e^{ikx}, \quad \hat{\Psi}^\dagger(x) = \sum_k \hat{\alpha}_k^+ e^{-ikx} \rightarrow \hat{\Psi}(x)\hat{\Psi}^\dagger(x) = \sum_k \hat{\alpha}_k^+ \hat{\alpha}_k^-
\]

Hence,

\[
\hat{H} = \hbar q \hbar_i \sum_k \hat{\alpha}_k^+ \hat{\alpha}_k^- \rightarrow |H| = \hbar q \sum_k |\hat{\alpha}_k^+||\hat{\alpha}_k^-| \approx \frac{\hbar q}{2} \sqrt{p^4 - (2p\hbar x)^2}
\]
8.6 Concluding Remarks

Table 8.6-1 compares quantum mechanics to image theory. Quantum mechanical states are represented by orthonormal vectors, where the inner product of any two such distinct vectors vanishes. By contrast, image theory replaces orthonormal state vectors with complex numbers.

Observables in quantum mechanics are represented by Hermitian operators. When a Hermitian operator acts on a state vector, it returns the state vector along with a real eigenvalue. The eigenvalue represents a possible value for an observable if the quantum system is in a certain state.

<table>
<thead>
<tr>
<th>Comparing Quantum Theory to Image Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>Concept</td>
</tr>
<tr>
<td>States</td>
</tr>
<tr>
<td>Observables</td>
</tr>
<tr>
<td>Measurables</td>
</tr>
<tr>
<td>Quantization</td>
</tr>
<tr>
<td>Uncertainty</td>
</tr>
<tr>
<td>Includes Gravity</td>
</tr>
</tbody>
</table>

Table 8.6-1

By contrast, an observable in image theory is described by projecting the state of the system into instantiation space. This is the analog of “taking a measurement” or “making an observation” in quantum mechanics. Numbers in instantiation space are complex.

‘Measurement’ in quantum mechanics reveals itself in two ways: 1) if two operators commute, then the values of the observables associated with each operator can be known simultaneously, 2) if two operators do not commute, then the values of the observables cannot be known simultaneously, but only probabilities for their values can be given. In other words, non-commuting operators have no common eigenvalues.

In image theory, a ‘state’ must be ‘instantiated’. If there is no interference in the instantiation, the “state” can be known. This is the analog of simultaneous measurement in quantum mechanics. On the other hand, interference represents a conflict between the state and its instantiation. For example, suppose the system has momentum ‘p’ and position ‘x’. In image theory this would be represented by the complex pair ‘(x,p)’. If the position ‘x’ is to be measured directly, when a measurement is taken, then

\[
\langle x, p \rangle \rightarrow \langle \sqrt{x}, \sqrt{p} \rangle \sqrt{x + p} = \sqrt{x^2 + px} \sqrt{p^2 + px},
\]
the interference term ‘\( p_x \)’ arises through the process of ‘instantiation’.

Both quantum mechanics and image theory support quantization. In each case, energy levels come in discrete quantities. But the physics is not quite the same, primarily because \( h \), ‘Planck’s constant’, plays a different role in each theory. The quantum mechanical \('h'\) is a constant of proportionality, where energy levels come in multiples of \( h \). But, in image theory, \( h \) is an instantiated limit. It is also the square of the quantum mechanical \('h'\) and represents something actually physically jiggling. Its dimensions are \([cycles/T]\).

State vectors in quantum mechanics are associated with a probability, where uncertainty exists in the state of a quantum system at all times. Image theory also promotes uncertainty, but of a different kind. Accordingly, the ontology of image theory contends that information comes to an observer incompletely and sometimes confusedly. Hence, the data collected concerning the nature of the Universe does not convey the objective reality of the thing observed. In this sense, image theory agrees with quantum mechanics, which generally forbids a deeper understanding of the Universe beyond predicting the results of a measurement. However, image theory does not preclude an understanding of the Universe at a deeper level, although such an understanding requires deeper insight than provided by direct observation.

Finally, quantum mechanics does not include gravity. Image theory not only includes gravity, but unifies the physics on the large-scale with physics on the small-scale. An analysis of this “unification” is given in the next chapter.
Chapter 9

EPILOGUE

THE NATURE OF THE UNIVERSE

“The more clearly we can focus our attention on the wonders and realities of the Universe about us, the less taste we shall have for destruction.”

— Rachel Carson

9.0 Introduction

Discovering the mysteries underlying the nature of the Universe is an exclusively human endeavor. Whether the Universe is a gift from God, a product of natural selection in a sea of random occurrences, or just one of many universes, understanding the nature of Universe is, at minimum, a vital human need, and, at the utmost, critical to sustaining human life. Proof of this plays out on religious days across the world as people pay homage to their “Creator”. From a practical perspective, an assumption is made that a greater understanding of the world’s laws leads to a better appreciation of what is and is not under human control, thereby offering insights on how to live within the dictates of those constraints. The yearning for understanding the conditions of human existence, and the importance of such insight, plays out daily in the drama of world politics and in the execution of war over professed ideologies.

Prospective on this topic has evolved over time, exemplified in tales of human triumph and tragedy. Such tales extol the virtues of understanding the laws of Nature and the consequences of willful ignorance. The Old Testament portrays a tale of the early Jews whose feeble intelligence was yet only able to grasp the simple concept of ‘obedience’. God lays down ten simple laws for life. Disobedience means hardship and ultimately destruction. The devil is portrayed as the enemy of God, someone who would replace God with himself and his own laws. The consequence of disobedience is death.

The New Testament portrays a different tale, where the Laws of God are “written on the hearts of men”. Humans are promoted to new intellectual vistas. God and Nature become one and the same. No longer obedient servants, humans control their fate. But the promotion comes with the increased responsibility of comprehending God’s laws at a deeper level and to act in accordance with Natural law. God bequeaths upon his human creations the gift of ‘reason’. Developing reason creates a path to human happiness, salvation and continued existence. Failure to grow rationally, to follow what is written on the human heart by God, results in destruction and death.

Evolution of the thinking on this subject does not end there as foretold in Erich Fromm’s epic tale “You Shall be as Gods”. Here humans reach the pinnacle of freedom and become the very object of their desire. Reaching their full potential means removing the
shackles procreated by fear of punishment and guilt. Human understanding of Nature completes itself. Human precepts align with the laws of Nature and man becomes one with God. Humans have yet to reach such lofty heights. Our present course aims more toward the destiny of the Dinosaurs, but the fact that such an exalted fate is even contemplated foretells the highest of human ideals and aspirations.

According to current interpretations, the Universe is comprised of space-time and its contents, consisting mostly of energy in its various forms, including electromagnetic radiation and matter, and therefore, planets, moons, stars, galaxies and the contents of intergalactic space [276]. The Universe also includes the physical laws that influence energy and matter, such as conservation laws, quantum mechanics, classical mechanics and relativity. Some philosophers and scientists support the inclusion of ideas and abstract concepts – such as mathematics and logic – in the definition of the Universe [277]. While these precepts explain what the Universe entails, no mention is made of the methods necessary for understanding the laws of the Universe and the importance of that understanding.

The scientific philosophy maintains that the best way of acquiring an understanding of the Universe is through direct observation. By collecting essential data, various hypotheses can be made into a theory, which is a set of rules describing a given phenomenon. Theories are subjected to the scrutiny of experimentation and observation. A hypothesis can be made that a more accurate set of data will match the theory more closely. Any significant deviation from carefully designed experiments and/or observations provides evidence that the theory needs modification.

The attraction to the scientific method is its great success. The scientific approach seems to work. The method has led to numerous discoveries that have made life easier and contributed significantly to the understanding of the laws of life. But even the most ardent supporter of the scientific approach must concede that much of what constitutes the Universe is not directly observable. In fact, ordinary matter comprises only about 5% of the constituents of the Universe. The rest is ‘dark’ matter and energy, so named because its substantial makeup is unknown.

It is almost universally agreed that mathematics is useful for formulating the laws of the Universe, even if an explanation for its usefulness remains elusive. There does not seem to be a direct connection between mathematics and the laws of Nature, at least, not one that can be undeniably verified. The usefulness of mathematics in describing the laws of the Universe comes as a great gift. For all intents and purposes, mathematics springs from logic. It could therefore be argued that the laws of the Universe should be logical, and therefore, understandable. But this does not seem to be a requirement. Quantum mechanics defies a logical explanation.

Image theory is principally an alternative approach to understanding the laws that govern the Universe. It employs unconventional methods that result in the unification of the theory of relativity with quantum mechanics. What follows is a summary of applying its methods to that end. The conclusion is that the Universe is an entity with a predisposition for existence. The rules of the Universe are eternal. They are life itself.
9.1 The Physics of the Big and the Small

Recall from chapter 4 that

\[ \sqrt{e} \left( \frac{1}{\sqrt{e}} \right)^{1-i\theta} = e^{i\theta} = \cos \theta + i \sin \theta \rightarrow \sqrt{e} \left( \frac{1}{\sqrt{e}} \right)^{1-i\pi \theta} = e^{i\pi \theta} = \cos(\pi \theta) + i \sin(\pi \theta) \]

Moreover,

\[ \frac{1}{\sqrt{e}} \left( \frac{1}{\sqrt{e}} \right)^{i\pi \theta - 1} \left( \sqrt{e} \right)^{-i\pi \theta} = e^{-i\pi \theta} = \cos(\pi \theta) - i \sin(\pi \theta) \]

Therefore, letting

\[ \Psi_b = \sqrt{e} \left( \frac{1}{\sqrt{e}} \right)^{1-i\pi \theta} \left( \sqrt{e} \right)^{i\pi \theta}, \quad \Psi_s = \frac{1}{\sqrt{e}} \left( \frac{1}{\sqrt{e}} \right)^{i\pi \theta - 1} \left( \sqrt{e} \right)^{-i\pi \theta} \rightarrow \Psi_b \Psi_s = e^{i\pi \theta} e^{-i\pi \theta} = 1 \]

The functions \( \Psi_b \) and \( \Psi_s \) are inverses of each other.

9.1.1 Small-Scale Physics

From Sec. 4.8, the form of the equation representing physics on the small-scale is

\[ \left( -\frac{1}{\sqrt{e}} \right)^{-1+\theta} \left( \sqrt{e} \right)^{-\theta} = -\sqrt{e} \left( 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \cdots \right) = -\sqrt{e} e^y = -\sqrt{e} e^{(-1+i\pi)\theta} \]

\[ \rightarrow \left( -\frac{1}{\sqrt{e}} \right)^{-\theta} \left( \sqrt{e} \right)^{-\theta} = e^{(-1+i\pi)\theta} = e^{-\theta} e^{i\pi \theta} = \left( \sqrt{e} \right)^{-\theta} \left( \sqrt{e} \right)^{-\theta} e^{i\pi \theta} \rightarrow \left( -\frac{1}{\sqrt{e}} \right)^{-\theta} \]

\[ = \left( \sqrt{e} \right)^{-\theta} e^{i\pi \theta} \rightarrow \left( -\frac{1}{\sqrt{e}} \right)^{-\theta} \left( \sqrt{e} \right)^{-\theta} = (-1)^{-\theta} = e^{i\pi \theta} = \cos(\pi \theta) + i \sin(\pi \theta), \]

\[ y = (-1 + i\pi) \theta \]

9.1.2 Large-Scale Physics

Similarly, the form of the equation representing physics on the large-scale is given by

\[ \left( -\frac{1}{\sqrt{e}} \right)^{1-\theta} \left( \sqrt{e} \right)^{\theta} = -\frac{1}{\sqrt{e}} \left( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \right) = -\frac{1}{\sqrt{e}} e^x = -\frac{1}{\sqrt{e}} e^{(1-i\pi)\theta} \]

\[ \rightarrow \left( -\frac{1}{\sqrt{e}} \right)^{-\theta} \left( \sqrt{e} \right)^{\theta} = e^{(1-i\pi)\theta} = e^{\theta} e^{-i\pi \theta} = \left( \sqrt{e} \right)^{\theta} \left( \sqrt{e} \right)^{\theta} e^{-i\pi \theta} \rightarrow \left( -\frac{1}{\sqrt{e}} \right)^{-\theta} \]

\[ = \left( \sqrt{e} \right)^{\theta} e^{-i\pi \theta} \rightarrow \left( -\frac{1}{\sqrt{e}} \right)^{-\theta} \left( \sqrt{e} \right)^{-\theta} = (-1)^{-\theta} = e^{-i\pi \theta} = \cos(\pi \theta) - i \sin(\pi \theta), \]

\[ x = (1 - i\pi) \theta \]
9.1.3 Combining the Small with the Large

Consider

\[ e^{i2\pi \theta} = \cos(2\pi \theta) + i \sin(2\pi \theta) \]

Note that this is the equation of a complex circle with period ‘1’. The square root of \( e^{i2\pi \theta} \) is

\[ (-1)^\theta = e^{i\pi \theta} = \cos(\pi \theta) + i \sin(\pi \theta) \]

Projection into instantiation space gives

\[ e^{i2\pi \theta} = \langle \cos(\pi \theta), \sin(\pi \theta) \rangle \sqrt{\cos^2(\pi \theta) + \sin^2(\pi \theta)} = \cos(\pi \theta) + i \sin(\pi \theta) = (-1)^\theta, \]

\[ \cos^2(\pi \theta) + \sin^2(\pi \theta) = 1 \]

Note that the complex circle \( e^{i2\pi \theta} \) is instantiated by a complex circle with period ‘2’. Hence, the points \( \langle x, y \rangle \) on the circle \( e^{i2\pi \theta} \) are instantiated by the points \( \langle \bar{x}, \bar{y} \rangle \) on the semicircle \( e^{i\pi \theta} \) (see fig. 9.1.3-1).

On the other hand,

\[ e^{-i2\pi \theta} = \cos(2\pi \theta) - i \sin(2\pi \theta) \]

is also a complex circle with period ‘1’. The square root of \( e^{-i2\pi \theta} \) is

\[ (-1)^{-\theta} = e^{-i\pi \theta} = \cos(\pi \theta) - i \sin(\pi \theta) \]

Projection into instantiation space gives

\[ e^{-i2\pi \theta} = \langle \cos(\pi \theta), -\sin(\pi \theta) \rangle \sqrt{\cos^2(\pi \theta) + \sin^2(\pi \theta)} = \cos(\pi \theta) - i \sin(\pi \theta) = (-1)^{-\theta}, \]

\[ \cos^2(\pi \theta) + \sin^2(\pi \theta) = 1 \]
In this case, the complex circle $e^{-i2\pi\theta}$ is instantiated by $e^{-i\pi\theta}$, a complex circle with period '2'. Hence, the points $(x, y)$ on the circle $e^{-i2\pi\theta}$ are instantiated by the points $(\bar{x}, \bar{y})$ on the semicircle $e^{-i\pi\theta}$ (see fig. 9.1.3-2).

Hence, for small scale physics,

\[
\left(\frac{-1}{\sqrt{e}}\right)^{1+\theta} (\sqrt{e})^{-\theta} = -\sqrt{e}e^y \to \left(\frac{-1}{\sqrt{e}}\right)^{-\theta} (\sqrt{e})^{\theta} e^{(i\pi-1)\theta} = 1 \to \left(\frac{-1}{\sqrt{e}}\right)^{-\theta} (\sqrt{e})^{\theta} e^{i\pi\theta} e^{-\theta} = (\frac{-1}{\sqrt{e}})^{-\theta} (\sqrt{e})^{\theta} e^{i\pi\theta} e^{-\theta} = (1)^{-\theta} e^{i\pi\theta} = 1
\]

Since $e = G \times 10^{15}$, then

\[
(-1)^{-\theta}(G \times 10^{15})^{i\pi\theta} = 1 \to (G \times 10^{15})^{i\pi\theta} = (-1)^{\theta} \to \ln(G \times 10^{15})^{i\pi\theta} = \ln(-1)^{\theta} \to i\pi\theta \ln(G \times 10^{15}) = \theta \ln(-1) = i\pi\theta \to \ln(G \times 10^{15}) = \ln G + 15 \ln 10 = 1
\]

This should be plain since $\ln e = \ln(G \times 10^{15}) = 1$.

Let

\[
\Psi_s = (G \times 10^{15})^{i\pi t} \to \frac{(-1)^{-t}\Psi_s}{8\pi y^2\sqrt{G}} = -\frac{\ln G + 15 \ln 10}{8\pi y^2\sqrt{G}} \to \Psi_s = (\ln G + 15 \ln 10)(-1)^t, \\
\gamma^2 = \frac{1}{1 - G^2t^2}, \quad c = 1,
\]

where $\Psi_s$ is called the ‘wave function’ on the small-scale. Note that $\Psi_s$ is a complex circle of period '2'.
On the large-scale,

\[
\left( \frac{-1}{\sqrt{e}} \right)^{1-\theta} (\sqrt{e})^\theta = -\frac{e^x}{\sqrt{e}},
\]

\[
x = (1 - i\pi)\theta \to (-1)^t (G \times 10^{15})^{-int} = 1 \to -8\pi y^2 \sqrt{G} (-1)^t (G \times 10^{15})^{-int}
\]

\[
= -8\pi y \sqrt{G} (\ln G + 15 \ln 10),
\]

\[
\Psi_b = (G \times 10^{15})^{-int} = (\ln G + 15 \ln 10) (-1)^{-t},
\]

where \(\Psi_b\) is called the 'wave function' on the large-scale. Note that

\[
\Psi_b \Psi_s = 1
\]

In the previous chapter, a gravitational tensor equation was developed. Letting

\[
\kappa(y) = -8\pi y^2 \sqrt{G}, \quad y^2 = \frac{1}{1 - G^2 t^2},
\]

then

\[
R_{\mu\nu} = \kappa(y) T_{\mu\nu}, \quad \mu, \nu = 0, ..., 3, \quad c = 1
\]

Recall that the equation above is the modified version of Einstein’s equations of gravity.

### 9.1.3.1 A Tensor Equation for the Small-Scale

A comparable tensor equation for small-scale physics is mathematically feasible, but would preclude a universe without matter and energy. In such a universe, flat spaces would be eliminated, including Ricci flat spaces, since \(T_{\mu\nu} \neq 0\). Moreover, if \(T_{\mu\nu}\) is to have an inverse, the elements of the \(T_{\mu\nu}\) must be functions with continuous second order derivatives, \(T_{\mu\nu} = T_{\nu\mu}\), \(\det T_{\mu\nu} \neq 0\) and \(T_{\mu\nu}\) must be a tensor.

To begin, recall that the square of the proper-time element ‘\(d\tau\)’ is designated

\[
d\tau^2 = \frac{1}{c^2} g_{\mu\nu} dx^\mu dx^\nu \to d\tau^2 = \frac{1}{c^2} g^{\mu\nu} dx_\mu dx_\nu, \quad g^{\mu\nu} g_{\mu\nu} = \delta^\mu_\nu
\]

To see this, recall that \(g_{ij}\) is a tensor, which has \(n^2\) elements and can be represented by an \(n \times n\) matrix. If \(g = \det g_{jk} \neq 0\), define

\[
g_{jk} = \frac{\text{cofactor of } g_{jk}}{g}
\]

Let the cofactor of \(g_{jk}\) be denoted ‘\(G(j, k)\)’. If \(n = 3\), by the theory of determinants,

\[
G(2, 1) = (-1)^{2+1} \begin{vmatrix} g_{12} & g_{13} \\ g_{32} & g_{33} \end{vmatrix}, \quad G(2, 2) = (-1)^{2+2} \begin{vmatrix} g_{11} & g_{13} \\ g_{31} & g_{33} \end{vmatrix}, \quad G(2, 3) = (-1)^{2+3} \begin{vmatrix} g_{11} & g_{12} \\ g_{31} & g_{32} \end{vmatrix}
\]

198
Hence,
\[ g_{21}G(2,1) + g_{22}G(2,2) + g_{23}G(2,3) = g \]
This procedure can be extended to \( n \) dimensions. In general,
\[ g_{jk}G(p,k) = 0, \quad \text{if } j \neq p \]
Since
\[ g_{jk} = \frac{\text{cofactor of } g_{jk}}{g} = \frac{G(j,k)}{g}, \]
then
\[ g_{jk}g_{jk} = g_{jk} \frac{\text{cofactor of } g_{jk}}{g} = g_{jk} \frac{G(j,k)}{g} = \frac{g}{g} = 1 \]
In this case, the summation is over \( k \) only. Note that \( g_{jk}g^{pk} = 0, \text{ if } p \neq j \). Hence, \( g_{jk}g^{pk} = \delta^p_j \). Since \( \delta^p_j \) is a tensor, \( g^{pk} \) is a tensor.

The proper-time element
\[
d\tau^2 = \frac{1}{c^2} g_{\mu\nu} d\dot{x}^\mu d\dot{x}^\nu \rightarrow \tau = \int \sqrt{\epsilon g_{\mu\nu} d\dot{x}^\mu d\dot{x}^\nu} \, dt, \quad \epsilon = \begin{cases} 1, & \text{if } g_{\mu\nu} d\dot{x}^\mu d\dot{x}^\nu \geq 0 \\ -1, & \text{if } g_{\mu\nu} d\dot{x}^\mu d\dot{x}^\nu < 0 \end{cases} \]
c = 1,
where \( \tau \) is a 'scalar', and hence, invariant. Suppose
\[ \int \sqrt{\epsilon g_{\mu\nu} d\dot{x}^\mu d\dot{x}^\nu} \, dt = T, \]
then let
\[ \tau^{-1} = \frac{1}{T^2} \int \sqrt{\epsilon g_{\mu\nu} d\dot{x}^\mu d\dot{x}^\nu} \, dt = \frac{1}{T} \]
if, in fact, \( \tau^{-1} \) exists. Since \( \tau \) is invariant, if \( \tau = 0 \) in one coordinate system, then \( \tau = 0 \) in all allowable coordinate systems. Hence, for \( \tau^{-1} \) to exist, \( \tau \neq 0 \), where \( \tau^{-1} \) might be called the 'proper frequency'.

Recall the Riemann-Christoffel curvature tensor:
\[
R^\alpha_{\mu\nu\beta} = \frac{\partial \Gamma^\alpha_{\mu\beta}}{\partial x^\nu} - \frac{\partial \Gamma^\alpha_{\nu\beta}}{\partial x^\mu} + \Gamma^\alpha_{\nu\delta} \Gamma^\delta_{\mu\beta} - \Gamma^\alpha_{\nu\beta} \Gamma^\delta_{\mu\delta} \rightarrow dV^\mu = R^\mu_{\sigma\nu\tau} dx^\sigma dx^\nu V^\tau,
\]
where $R_{\mu\nu\beta}^\alpha$ is made up of the Christoffel symbols and their derivatives, which, in turn, consist of the $g_{\mu\nu}$'s and their derivatives and $dV^\mu$ represents the change in the vector 'V' as it is parallel transported around a closed loop.

Note that

$$g^{\alpha\beta}R_{\mu\nu\beta}^\alpha = R_{j\mu\nu\beta}^k \rightarrow g_{\mu\nu}R_{j\mu\nu\beta} = R_{j\mu\nu\beta}^k \rightarrow g_{\beta m}R_{j\mu\nu\beta} = R_{j\mu\nu\beta}^k \rightarrow dV_{\mu}$$

Evidently then,

$$(dV^\mu)(dV_\mu) = dV^2$$

is a scalar, and hence, invariant. But since $dV^\mu$ is a tensor, if it is zero in one coordinate system, it is zero in all allowable coordinate systems. Therefore,

$$dV^\mu = 0 \rightarrow dV_\mu = 0 \rightarrow dV = 0$$

For $dV^{-1}$ to exist, $dV^\mu \neq 0$ (no Euclidean flat spaces).

It would be desirable to find a tensor $S_{\alpha}^{\mu\nu\beta}$ such that

$$R_{\mu\nu\beta}^\alpha S_{\tau}^{\mu\nu\beta} = \delta_\tau^\alpha,$$

making the Riemann-Christoffel tensor invertible. For a tensor to be invertible, it must be symmetric in the indices. However, the Riemann-Christoffel tensor does not have this property. However, the Ricci tensor, which is a contraction on indices 'a, v' of the Riemann-Christoffel tensor, is symmetric in the indices, and therefore, there is the possibility of inverting it. According the Levi-Civita method,

$$S_{\mu\nu} = 4 \frac{R_{\kappa\tau}R_{\lambda\rho}R_{\xi\sigma}e^{\mu\kappa\lambda\xi}e^{\nu\pi\rho\sigma}}{R_{\alpha\xi}R_{\beta\eta}R_{\gamma\theta}e^{\alpha\beta\gamma\delta}e^{\epsilon\eta\theta\tau}} \rightarrow R_{\mu\alpha}S_{\alpha\nu} = S_{\nu\alpha}R_{\alpha\mu} = \delta_\nu^\mu$$

Under these circumstances, it becomes possible to write a tensor equation for small-scale physics:

$$S_{\mu\nu} = -\frac{V_{\mu\nu}}{8\pi\gamma^2\sqrt{G}}, \quad \gamma^2 = \frac{1}{1 - G^2 t^2}, \quad R_{\mu\alpha}S_{\alpha\nu} = S_{\nu\alpha}R_{\alpha\mu} = \delta_\mu^\nu, \quad c = 1,$$

Conceived in this way, the Universe becomes a giant harmonic oscillator. The factor ‘$-8\pi\gamma^2\sqrt{G}$’ is small, if $t$ is small. So, in the beginning, the Universe was a mass of frenzied vibrational energy on a small-scale. However, as time advances, the large-scale, which is virtually without energy in the beginning, gains energy at the expense of the small-scale. This process continues for a time, but, at some point, $\gamma^2$ must reverse direction. At such time, the large-scale begins losing energy while the small-scale gains energy once again.
In the beginning, the large-scale is mostly void, but, as time advances, it gains energy. Conversely, on the small-scale, there is frenzied activity, signifying a massive amount of energy. The process is illustrated in fig. 9.1.3.1-1 a and b.

![Figure 9.1.3.1-1](image_url)

**9.2 The Time Epoch of the Universe**

At \( t \approx 0 \), the large-scale is virtually without energy, while the small-scale is at maximum energy. As time advances, the large-scale gains energy at the expense of the small-scale. As \( t \to 1 \), which is called a '1/2-epoch', the energy at the large-scale is a maximum, while the energy at the small-scale is a minimum. At this point, the process reverses. As \( t = 1 \to 0 \), the small-scale gains energy at the expense of the large-scale, suggesting that the life of the Universe is time-cyclic:

\[
\approx 0 \to \approx 1 \to \approx 0 \to \approx 1 \to \approx 0 \ldots
\]

Therefore, \( \gamma \) should be a cyclic function of time.

**9.2.1 The Wave Length of the Universe**

Remembering that

\[
d\tau = \frac{dt}{\sqrt{1 - \frac{\bar{v}(t)^2}{c^2}}} \to d\tau = \frac{dt}{\sqrt{1 - \frac{[l]^2G^2c^4}{c^2}}} \to d\tau^2 = \frac{dt^2}{1 - \frac{[l]^2G^2c^4}{c^2}} \to \frac{d\tau^2}{dt^2} = \frac{1}{1 - \frac{[l]^2G^2c^4}{c^2}},
\]

where \( d\tau \) is called the ‘cosmic proper-time’ element and \( dt \) is called the ‘arbitrary’ or ‘coordinate’ time element. If \( G,c \) is chosen such that \( G,c = 1 \), then

\[
d\tau = \frac{dt}{\sqrt{1 - t^2}}
\]
If cosmic proper-time is to be cyclic, then \( t \) should reach its maximum at \( t = 1/2 \), giving an occasion for the cycle to reverse itself so that the time \( t' \) goes from \( 1/2 \rightarrow 0 \). Hence, in the equations above, \( t \) should be replaced by \( 2t \). Figure 9.1.3.1-1 illustrates this.

![Cosmic Time vs. Scale](image)

Figure 9.2.1-1

Now

\[
\frac{G^2}{c^2} \approx 8.21 \times 10^{-51} \rightarrow 4t_{\text{max}}^2 \approx \frac{1}{8.21} \times 10^{51} \rightarrow 2t_{\text{max}} = \sqrt{\frac{1}{8.21} \times 10^{51}} \approx 1.1 \times 10^{25} \text{ [sec]}
\]

\[ \rightarrow t_{\text{max}} \approx 5.5 \times 10^{24} \text{ [sec]} \]

As a conjecture for a cyclic function of time, let

\[
\frac{1}{\gamma} \approx \frac{1}{1 - \cos^2(\lambda \pi t)}, \quad \lambda = \frac{G}{c}
\]

To simplify matters, choose dimensional units such that \( G, c = 1 \), then

\[
\frac{1}{\gamma} \approx \frac{1}{\sin^2(\pi t)} \rightarrow \gamma = \sin^2(\pi t)
\]

The graphs of \( \gamma \) and \( 1/\gamma \) are shown in figure 9.2.1-2 a, b. Evidently, the Universe cycles through epoch periods that last about \( 10^{25} \text{ [sec]} \), which is approximately \( 3.2 \times 10^{17} \text{ years} \) or about \( 3.2 \times 10^8 \text{ billion years} \).
The model shown in fig. 9.2.1-2 is not entirely satisfactory. For one thing, the small-scale physics encounters a singularity at the epochs. Secondly, at the epoch and $1/2$-epoch periods, the derivative of $\sin^2(\pi t)$ is zero. Hence, $d(1/\gamma)/dt$ is undefined at those points.

### 9.3 Deriving the Wave Function of the Universe

However, the model can be improved if the large- and small-scale equations are reinterpreted. For example, as an estimate, let

$$\frac{1}{\gamma} \approx \frac{1}{\lambda - \cos^2(\pi t)} \rightarrow \gamma = \lambda - \cos^2(\pi t),$$

where '$\lambda = 1.001$' is slightly greater than 1 and $1/\lambda = 1/1001$ is slightly less than 1 so that

$$\lambda - \frac{1}{\lambda} = 1.001 - \frac{1}{1.001} \neq 0$$

Let

$$\frac{d\gamma}{dt} = \lambda - \cos^2(\pi t) \rightarrow \gamma = \int \lambda - \cos^2(\pi t) \, dt$$

The graphs in fig. 9.2.1-2 are then the slopes of the tangent lines, which is now interpreted as the change in the energy with respect to time, which is never zero if

$$\lambda = 1 + \varepsilon, \quad \varepsilon \ll 1, \quad \varepsilon > 0$$
Let
\[ \varepsilon = \frac{G}{c} \to \frac{dy}{dt} = 1 + \varepsilon - \cos^2(\varepsilon \pi t) = \varepsilon + \sin^2(\varepsilon \pi t) \]

The following observations can be made: 1) the model shows a huge change in the vacuum energy at the epochs, consistent with what quantum theory predicts or, at least, should predict, 2) there are no negative energy states, 3) the change in energy on the small-scale is large at small distances, but dies out quickly at longer distances, also consistent with what quantum mechanics predicts, 4) energy is not conserved on the large-scale nor on the small-scale, but taken together, energy is conserved.

9.3.1 The Energy Interpretation of the Wave Function

An interpretation of how the Universe behaves can be given in terms of ‘cosmic time’. ‘Cosmic time’ means that the “rate of time” is determined by Nature. Whereas ‘arbitrary time’ implies that the “rate of time” is chosen arbitrarily. Let \( \tau \) be called ‘cosmic time’ and \( t \) called ‘arbitrary time’. The equation
\[ \frac{d\tau}{dt} = 1 + \sin^2(\pi t) \to \frac{1}{\left( \frac{d\tau}{dt} \right)} = \frac{1}{1 + \sin^2(\pi t)}, \quad \frac{d\tau}{dt} \frac{1}{\left( \frac{d\tau}{dt} \right)} = 1, \quad G, c = 1 \]
establishes a relationship between ‘cosmic’ and ‘arbitrary’ time (see fig. 9.3.1-1), where \( d\tau/dt \) is the change in ‘cosmic time’ with respect to ‘arbitrary’ time.

![Cosmic Time](image1)

![Cosmic Time](image2)

**Figure 9.3.1-1**

Regarding dimensions, let
\[ \frac{d\tau}{dt} \propto \frac{[l]G}{c} + h_i \sin^2 \left( \frac{[l]G\pi t}{c} \right), \quad [l] = \left[ \frac{cm}{\text{sec}^2} \right] \]
The factor \( h_i \) appears because it has dimensions \('[1/T]'\). Hence, the right-side of the equation above has dimensions \('[1/T]'\). The equation above is not satisfactory, since the left-side of the equation is dimensionless while the right-side has dimensions \('[1/T]'\), which forces the introduction of a factor \( \nu \) with dimensions \('[1/T]'\) so that the equation above becomes

\[
\frac{dt}{d\tau} = \frac{[l]G}{\nu c} + \frac{h_i}{\nu} \sin^2 \left( \frac{[l]G\pi t}{c} \right), \quad [l] = \left[ \frac{cm}{sec^2} \right], \quad \nu = 1
\]

However, \( h_i \) is much too small to provide a reasonable description of the phenomenon being postulated here. Moreover, it would be more useful if the equation was specified in terms of the energy, since modern physics is primarily concerned with the energy of a system. Consider

\[
\frac{dE}{dt} \propto \frac{G^2}{c^2} + h^{-2} \sin^2 \left( \frac{[l]G\pi t}{c} \right), \quad [l] = \left[ \frac{cm}{sec^2} \right],
\]

where \( \frac{dE}{dt} \) is the change in energy with respect to arbitrary time. Recall from chapter 5, there was a gravitational energy factor derived i.e.

\[
h^{-1} = \sqrt{\overline{v}_B^2 - \overline{v}_L^2} = 1.182257276 \times 10^{26} \left[ \frac{cm}{sec} \right] \rightarrow h^{-2} \approx 1.4 \times 10^{52} \left[ \frac{cm}{sec^2} \right]
\]

where \( \nu^2 \) is proportional to the energy of a system. The factor \( h^{-1} \) is given dimensional units \('[cm/\sec]'\). Since \( G \) is dimensionless, there is some flexibility in the assignment of dimensions. Let \([k]G\) have dimensional units \('[cm^4/sec^4]'\). Introduce a constant of proportionality \( \nu \) with dimensional units \('[gm/sec]'\). The terms on the right-side of the equation possess dimensional units \('[gm - cm^2/sec^3]'\) and the equation can be written

\[
\frac{dE}{dx} = [k]G^2 \nu + \nu h^{-2} \sin^2([l]G\pi x) \rightarrow E = \int [k]G^2 \nu + \nu h^{-2} \sin^2([l]G\pi x) \, dx + C,
\]

\[
C = \text{a constant}, \quad [k] = [L^4/L^4], \quad [\nu] = [M/L],
\]

where the modern view has been adopted that regards distance = time through the equation

\[
x = ct, \quad c = 1 \rightarrow dx = dt
\]
Although not to scale, the locus of \( \frac{dE}{dx} \) is shown in fig. 9.3.1-2:

![Graph](image1)

Figure 9.3.1-2

Hence,

\[
E = \int [k]G^2v + nh^{-2} \sin^2([l]G\pi x) \, dx + C \rightarrow E = \frac{3\pi x - \cos(\pi x) \sin(\pi x) + 2\pi}{2\pi},
\]

\( h^{-1}, c, G, v, C = 1 \)

(see fig. 9.3.1-3).

![Graph](image2)

Figure 9.3.1-3

Note that the equation above is consistent with Einstein’s equation, where

\[
E = mc^2 \rightarrow E = m, \quad c = 1
\]
To be clear,
\[
\frac{1}{dE} \rightarrow \frac{1}{\int dE} = \frac{1}{dx} = \frac{1}{E}
\]

### 9.4 Is the Universe Finite or Infinite?

Speculatively, recall the earlier relationships

\[
ph_i\lambda = E, \quad E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \rightarrow mvh_i\lambda = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}
\]

Replacing the mass ‘m’ on left-side of the equation above with the relativistic mass ‘m/\sqrt{1 - v^2/c^2}’, then

\[
\frac{mvh_i\lambda}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \rightarrow vh_i\lambda = c^2
\]

Since the mass cancels on either side of the equation, set \(v = c\), then

\[
h_i\lambda = c \rightarrow \lambda = \frac{c}{h_i} \approx 4.22 \times 10^{62} \text{ [cm]} \rightarrow \frac{\lambda}{c} = \frac{1}{h_i} \approx 1.4 \times 10^{52}
\]

Now

\[
x = h_i\lambda t, \quad h_i\lambda = v = c \rightarrow t = \frac{x}{v}
\]

Evidently, \(t = x/v\) is equal to the phase. Hence,

\[
\frac{x}{c} = h_i \frac{2\lambda}{c} t_{max} \rightarrow x = 2ct_{max} \approx 3.3 \times 10^{35} \text{ [cm]},
\]

which suggests that the large-scale universe is finite, but will expand to a maximum diameter of \(\approx 3.3 \times 10^{35} \text{ [cm]}\), about \(3.5 \times 10^{17} \text{ [light years]}\). The average temperature of the Universe at this time will be extremely small, representing the condition of minimum energy. Virtually no energy remains for further expansion. Shortly thereafter the Universe will begin another expansion period – a big bang.